New Microfoundations for the Aggregate Matching Function, with Empirical and Theoretical Implications

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Abstract

Although a number of different models have been suggested for the process that brings workers and firms together in the labour market, none of these performs well in empirical studies of the aggregate matching function. Empirically, the most successful functional form is Cobb-Douglas, for which there are no microfoundations in the existing literature. I present a new model for the matching process, based on a "telephone line" Poisson queuing process. This implies a CES matching function, approximately Cobb-Douglas when marginal search costs are approximately constant. The model provides an interpretation for empirical evidence, and insight into the theoretical efficiency conditions for matching models.

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The matching function is the lynchpin of search and matching models of the labour market. The rate of matching, *m*, is taken to be a function of the stock of unemployed workers, *u*, and the stock of vacant posts, *v*, in the market. If there were no frictions, matching would be instantaneous (and the number of matches would be determined by the short side of the market). But when workers and firms have to engage in a costly and time-consuming process of search to find each other, the matching function captures the technology that brings them together. It encapsulates search and matching frictions, allowing a more realistic description of the labour market, and of unemployment.

The matching function is by definition the source of frictional unemployment in such models; it may be the source of additional inefficiencies, due to search externalities, since each searching worker or firm affects the rate of matching for other agents. But the interpretation of existing theoretical results, and of empirical estimates, is limited by the assumption that the matching function is an exogenous "black box", unrelated to agents' behaviour or to other features of the labour market environment.

In this paper, I present simple microfoundations for the aggregate matching function. The model has several advantages: it can be directly integrated into standard search and matching models; it is consistent with, and guides the interpretation of, accumulated empirical evidence; and it provides some insight into the conditions for efficiency in search models.

1. Some Problems in the Literature on Matching Functions

There is a broad consensus in the literature on some desirable basic properties for the matching function $m(u,v)$. It should be increasing and concave in both arguments, with $m(0,v)= m(u,0)=0$. Constant returns is often imposed, and thought to be a reasonable assumption, although some models (for example, the well-known model of Diamond, 1982) use matching functions with increasing returns, resulting in a thin market externality and multiple equilibria.

Petrongolo and Pissarides (2001) provide a comprehensive survey of the matching function, focusing on microfoundations and empirical evidence. A variety of functional forms is suggested by different specifications of the matching process. One category of models is derived from static "urn-ball" processes: suppose, for example, that in each period a proportion α of unemployed workers each place a ball (job application) in a randomly chosen urn (a job vacancy), and the employer fills the vacancy by selecting a ball from the urn at random. Here, matching frictions are due to a co-ordination problem between workers resulting in congestion – some urns receive several balls and others none. The parameter α represents the search intensity of unemployed workers. Then the expected number of matches is equal to the number of urns with at least one ball:

$$m = v(1-(1-1/v)^{\alpha u}) \tag{1}$$

The matching function (1) can be approximated for large v (holding v/u constant) by:

$$m = v(1-\exp(-\alpha u/v)) \tag{2}$$

This is the standard urn-ball functional form, obtained by (among others) Hall (1979) and Peters (1984). For a continuous time version we could suppose that each worker sends out applications at constant Poisson rate α to randomly chosen vacancies. Then the expected number of vacancies receiving at least one application in a time period of length dt is $v(1-\exp(-\alpha u dt/v))$, as in the urn-ball model above. Letting dt tend to zero gives a Poisson matching rate that is linear in unemployment:

$$m = \alpha u \tag{3}$$

Matching functions (1) to (3) represent an asymmetric view of the matching process, in which all search is undertaken by workers, and firms are merely passive recipients of applications. We could extend the continuous time version of the model (as in Mortensen and Pissarides, 1999) to allow for simultaneous search on both sides of the market, so that firms separately make contact with workers at recruitment rate γ. This leads to a symmetric linear matching technology:

$$m = \alpha u + \gamma v \tag{4}$$

There are several alternative approaches: for example, stock-flow matching (Coles and Smith, 1998) in which the problem is heterogeneity and mismatch between the existing stocks of unmatched agents, rather than any coordination failure; and the statistical aggregation approach which is based on the assumption of limited mobility of capital and labour[1]. However, urn-ball models have an intuitive appeal, and remain popular in theoretical discussion when interpretation for the matching function is required (as in, for example, Acemoglu and Shimer, 1999).

Despite their popularity, matching functions derived from urn-ball models do not have particularly desirable theoretical properties and cannot easily be integrated into standard search models, most of which are continuous time models treating workers and firms symmetrically. The ubiquitous function (2), which does at least satisfy the basic requirements outlined at the beginning of this section, is derived from a static, asymmetric, model. In the continuous time equivalent (3) the main attractive feature of the urn-ball model has been lost: the probability that any vacancy receives more than one application is of order dt^2 so the congestion effect disappears in the limit. Furthermore, the linear technologies (3) and (4) do not behave well when either u or v is small relative to the other, and in particular do not satisfy the basic requirement that $m(0,v)= m(u,0)=0$.

Empirically, urn-ball models do not perform well. The most successful empirical functional form is Cobb-Douglas, for which microfoundations have never been provided. Petrongolo and Pissarides summarise the wealth of empirical support for a Cobb-Douglas matching function with constant returns to scale, and an elasticity with respect to unemployment of between 0.5 and 0.7. Blanchard and Diamond (1990) also estimated a CES function, obtaining an elasticity of substitution between unemployment and vacancies not significantly different from unity (corresponding to the Cobb-Douglas case). Estimates of the unemployment elasticity in European countries have tended to be higher than those for the US. The lack of microfoundations, however, makes it difficult to interpret such results – theory gives no explanation for a higher unemployment elasticity in one market than in another.

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[1] The matching function is derived by statistical aggregation of unemployment and vacancies in a set of micromarkets, where the short side of each market clears, but there is no mobility between micromarkets. A log-normal distribution of the vacancy-unemployment ratio across micromarkets implies a CES-type aggregate matching function.

A recurrent theoretical theme is the effect of search externalities, which are a by-product of the matching function, on the efficiency of the equilibrium. Hosios (1990) identified a general condition under which all externalities are internalised and all decisions are efficient: the elasticity of matching with respect to unemployment must be equal to the worker's share of the match surplus. More precisely, let the matching function be $m(u,v,\alpha,\gamma)$ where α and γ represent the search and recruitment intensities of workers and firms respectively. Assume that search effort is "input-augmenting" (Pissarides, 2000) so that the matching function can be written:

$$m(u,v,\alpha,\gamma) \equiv \tilde{m}(\alpha u, \gamma v) \tag{5}$$

and also that the match surplus is shared between worker and firm according to a Nash bargain with a share β for the worker. Then entry decisions, job acceptance and destruction decisions and choices of search and recruitment intensity α and γ are jointly efficient if and only if, firstly, the matching function has constant returns to scale and secondly, the unemployment elasticity is equal to the worker's bargaining share:

$$\frac{u}{m} m_u = \beta \tag{6}$$

The Hosios efficiency condition (6) is somewhat unsatisfactory, in that there is no particular reason why it should be satisfied, and yet nothing to explain why it is not. Suppose, for example, bargaining power β is constant, and the matching function is Cobb-Douglas so that the elasticity is also constant. Even in that case, as discussed by Pissarides (2000), there is no reason why they should be equal – since bargaining power is "determined in a different environment and without reference to the structural properties of the matching technology". Efficiency in the Cobb-Douglas case requires the equality of two exogenous parameters, and there is nothing in the model to suggest how they are determined, or any link between them. Pissarides argues, therefore, that we are unlikely to find intuition for the Hosios condition.

One approach to the resolution of this problem is to maintain the assumption of an exogenous black-box matching function, but consider alternatives to Nash bargaining for the determination of wages, and to look for conditions under which the resulting surplus shares satisfy the Hosios condition. Shimer (1996) and Moen (1997) show that if firms announce wage-contracts, and workers are sufficiently well-informed to direct their search towards firms posting particular contracts, the Hosios condition holds in equilibrium. In these models, the mechanism that leads to efficiency is that by varying the wage a firm can vary its flow of applicants – because workers are aware of such variations and direct their search accordingly. This creates effective competition between firms, which disciplines them to make wage offers that internalise search externalities. These models can be interpreted as an intermediate case between standard matching models in which workers are less well-informed, and perfect competition (Moen calls it "competitive search equilibrium"). However, they provide a solution to the problem of the Hosios condition only under the more stringent information conditions that allow directed search.

In what follows I present microfoundations for a very simple matching process. It is a continuous-time process that treats workers and firms symmetrically, so it can be easily integrated into standard search models. It can be seen as a continuous-time analogue of an urn-ball model, capturing the same kind of congestion problem between agents on the same side of the market. The resulting matching function is CES, and approximately Cobb-Douglas when marginal search costs are approximately constant. Hence it is consistent with existing empirical evidence, but also identifies the *determinants* of the unemployment elasticity, and of the elasticity of substitution between unemployment and vacancies, giving us an interpretation of empirical estimates of these parameters. Finally, since the unemployment elasticity is not an exogenous parameter, but depends on (amongst other things) the worker's bargaining share β, the model offers a different perspective on the Hosios efficiency condition.

2. The Matching Model

The framework for the matching model is the same as that of Hosios (1990) – the main innovation in this paper comes from the specification of a particular matching technology, and particular search cost functions. We have a mass of identical workers and a mass of identical firms, and each firm can employ one worker. Let u be the number of unemployed workers and v the number of vacant jobs. When an unemployed worker meets a firm with a vacancy, the flow productivity y of their match is realised, as a random variable from some continuous distribution $F(y)$. Provided that the net surplus from the match is positive, the match is consummated and the surplus is shared according to a Nash bargain in which the bargaining power of the worker is $\beta \in (0,1)$. Productivity remains constant for the duration of the match, but matches are destroyed at exogenous Poisson rate λ, in which case the worker re-enters unemployment, and the job becomes vacant.

All agents are risk-neutral, with positive[2] discount rate r. The flow rate of meeting[3] is given by a function $m(u,v,\alpha,\gamma)$. Unemployed workers and firms with vacancies choose their search and recruitment intensities α and γ to maximise their expected present value of income. We will assume that they have convex cost functions $C_w(\alpha)$ and $C_f(\gamma)$. In addition, workers obtain flow benefit z while unemployed, and firms incur a fixed flow cost x of maintaining a vacancy.

In a steady-state equilibrium let the expected present value of income for an unemployed worker be U, and the expected present value to the firm of a vacancy be V. For a match with productivity y and wage w, let the worker's valuation be W and the firm's valuation be J. Since matches break down at Poisson rate λ, these satisfy the Bellman equations:

$$\begin{aligned} rW &= w + \lambda(U - W) \\ rJ &= y - w + \lambda(V - J) \end{aligned} \tag{7}$$

Hence the expected net surpluses for the worker and firm are, respectively:

$$W - U = \frac{w - rU}{r + \lambda} \text{ and } J - V = \frac{y - w - rV}{r + \lambda} \tag{8}$$

A match is jointly acceptable if and only if it can provide a positive net surplus for both parties – that is, if it has productivity greater than or equal to the reservation value y^* defined by:

[2] Hosios (1990) presented his results for the limiting case of a zero discount rate, since this makes the derivation of the efficiency results much simpler. However, these results also hold for a positive discount rate (see for example Pissarides, 2000) so there is no need here to let r tend to zero.
[3] We follow Hosios (1990) in letting the matching function represent the rate of *meeting*, or the *contact rate* (Pissarides, 2000). Whether or not a meeting results in a consummated match depends on its realized productivity.

$$y^* \equiv r(U+V) \tag{9}$$

For a match with productivity $y \geq y^*$, the wage is determined according to a generalised Nash bargain: $w = rU + \beta(y - y^*)$. So the worker's and firm's expected surpluses from an acceptable match can be written:

$$W - U = \frac{\beta(y - y^*)}{r + \lambda} \text{ and } J - V = \frac{(1-\beta)(y - y^*)}{r + \lambda} \tag{10}$$

Ex-ante (before y is revealed), their expected gains from a contact are therefore βS and $(1-\beta)S$, where S is the expected total surplus:

$$S = \frac{1}{r + \lambda} E[y - y^* | y \geq y^*] \Pr[y \geq y^*] = \int_{y^*} \frac{y - y^*}{r + \lambda} dF(y) \tag{11}$$

Let α and γ be the equilibrium search and recruitment intensities, and consider a worker who sends out applications at current rate α_i (for a time interval of length dt) and at future rate α. The probability that an application makes contact with a firm is $(m/\alpha u)dt$, so his valuation of unemployment is:

$$U(\alpha_i) = (z - C_w(\alpha_i))dt + (1 - rdt)\left(U(\alpha) + \frac{\alpha_i m}{\alpha u} dt \beta S \right) \tag{12}$$

Maximising with respect to α_i and setting $\alpha_i = \alpha$ leads to the first-order condition for equilibrium search intensity:

$$C_w'(\alpha) = \frac{m}{\alpha u} \beta S \tag{13}$$

and setting $\alpha_i = \alpha$ and letting $dt \to 0$ in (12) determines the equilibrium valuation of unemployment:

$$rU = z - C_w(\alpha) + \frac{m}{u} \beta S \tag{14}$$

Similarly for the firm:

$$C_f{}'(\gamma) = \frac{m}{\gamma v}(1-\beta)S \qquad (15)$$

and

$$rV = -x - C_f(\gamma) + \frac{m}{v}(1-\beta)S \qquad (16)$$

Adding (14) and (16), using (11), gives an equation for the equilibrium reservation productivity y^*. Then the determination of the equilibrium is completed by imposing the steady-state condition that the inflow to jobs must equal the outflow to unemployment, together with entry conditions for workers and firms – typically that the mass of workers is fixed, and free-entry for firms.

3. The Matching Technology

The model for the matching technology is based on the classic "telephone line" Poisson queuing process (Cox and Miller, 1965). Workers send out applications (make calls) randomly to firms with vacancies at Poisson rate α, and the firms respond to applications at Poisson rate γ. This corresponds to workers sending applications at exponentially-distributed time intervals with expectation $1/\alpha$, and similarly firms taking an exponentially-distributed length of time to process an application, with expectation $1/\gamma$. We can interpret the processing time as the time required to ascertain the productivity of the match. If, when the worker makes a call to a particular vacancy, the firm is already engaged in processing another application, the line is busy and the call does not get through – so the application fails. This is the main technological assumption, which captures in a simple way the same kind of congestion or co-ordination problem as in urn-ball models of matching. When the firm finishes processing an application, it either enters into the match (if it is sufficiently productive) or rejects the application and waits for further calls. So at any instant, a vacancy is in one of two states: it is either "open" and waiting for applications, or busy processing.

The total number of vacancies is v; let v_o be the number of open vacancies. When u is the number of unemployed workers, the total number of applications sent out per unit time is αu, so the rate of arrival of applications at each vacancy is $\alpha u/v$, and the number of applications arriving at open vacancies and therefore entering the processing system is $\alpha u v_o/v$. In steady-state, this is equal to the outflow from application processing, which is $\gamma(v\text{-}v_o)$. Equating these two expressions and rearranging gives the proportion of vacancies that are open at any instant, in the steady state:

$$\frac{v_o}{v} = \frac{\gamma v}{\alpha u + \gamma v} \tag{17}$$

Since this expression is the probability of any individual application encountering an open vacancy, the total number of successful meetings per unit time – that is the matching rate – is given by:

$$m(u, v, \alpha, \gamma) = \frac{\alpha u \gamma v}{\alpha u + \gamma v} \tag{18}$$

This matching function is, as required, increasing and concave in u and v. It has constant returns to scale. It is also increasing and concave in the workers' job search intensity α, and in the firms' recruitment intensity γ, which are input-augmenting. The rate of matching tends to zero as u or v tends to zero, and as the number of vacancies tends to infinity the rate of matching tends to αu, which is the rate of applications. In short, the function has all of the properties we would like it to have. Note also that it is a CES function, with elasticity of substitution between unemployment and vacancies equal to a half.

Since this matching function has constant returns and input-augmenting search intensities, the Hosios condition (6) is necessary and sufficient for efficient decisions on three margins – entry, search and recruitment intensity, and job creation and destruction. The elasticity of matching with respect to unemployment is given by:

$$\eta \equiv \frac{\partial \log m}{\partial \log u} = \frac{\gamma v}{\alpha u + \gamma v} = \frac{m}{\alpha u} \tag{19}$$

which, from above, is the probability that an individual application makes successful contact with a firm, or equivalently the proportion of total search effort undertaken by firms. Thus, a high unemployment elasticity corresponds to a low congestion problem facing workers. Correspondingly this implies a high congestion problem facing firms, in the sense that their recruitment effort is high relative to applications, so individual firms face a lower contact probability.

4. The Unconditional Matching Function

The matching function (18) determines the matching rate $m(u,v,\alpha,\gamma)$ conditional upon search intensities α and γ. Equations (13) and (15) define (at least implicitly) the worker's and firm's optimally chosen search and recruitment intensities, $\alpha = \alpha^*(u,v)$ and $\gamma = \gamma^*(u,v)$. By substituting these back into the conditional matching function, we can obtain the matching function with endogenous search, or *unconditional matching function*:

$$m^*(u,v) \equiv m\left(u,v,\alpha^*(u,v),\gamma^*(u,v)\right) \tag{20}$$

It is the properties of this function m^* that are of empirical interest, since search and recruitment effort are not normally observed. The relationship between the two functions depends on the form of the search and recruitment cost functions. Suppose that they have constant and identical elasticities[4]:

$$C_w(\alpha) = \frac{c_w}{k}\alpha^k \text{ and } C_f(\gamma) = \frac{c_f}{k}\gamma^k \text{ where } k>1, \text{ and } c_w, c_f > 0 \tag{21}$$

Then, using (13) and (15) to eliminate α and γ from the definition of the matching function (18) gives:

$$m = (mS)^{1/k}\frac{uv\left(\dfrac{\beta}{c_w u}\right)^{1/k}\left(\dfrac{1-\beta}{c_f v}\right)^{1/k}}{u\left(\dfrac{\beta}{c_w u}\right)^{1/k} + v\left(\dfrac{1-\beta}{c_f v}\right)^{1/k}} \tag{22}$$

Defining:

$$\rho \equiv 1 - 1/k\,, \quad \overline{\eta} \equiv \frac{c_w/\beta}{c_w/\beta + c_f/(1-\beta)}, \text{ and } \overline{S} \equiv \frac{S}{c_w/\beta + c_f/(1-\beta)} \tag{23}$$

[4] In section 5 below we will generalize to allow for the two functions to have different elasticities.

and rearranging (22) to solve for *m*, gives after a little manipulation a CES form for the unconditional matching function:

$$m^*(u,v) \equiv \bar{S}^{\frac{1-\rho}{\rho}} \left(\frac{1}{\bar{\eta}^{1-\rho} u^{-\rho} + (1-\bar{\eta})^{1-\rho} v^{-\rho}} \right)^{\frac{1}{\rho}} \tag{24}$$

4.1 Results

We can summarise the implications of this analysis as follows. With endogenous search and recruitment intensities, and constant elasticity search cost functions (21), the matching process described in section 3 implies that:

(i) *The aggregate matching function is CES.*

The conditional matching rate *m* is given by equation (18); the unconditional matching rate *m** is a CES function of unemployment, *u*, and vacancies, *v*, given by equation (24).

(ii) *The elasticity of substitution between unemployment and vacancies is determined by the elasticity of search and recruitment costs.*

In equation (24), the elasticity of substitution, σ^*, between *u* and *v*, is given by $\sigma^* = 1/(1+\rho)$. Hence from the definition of ρ above it depends on the elasticity of the search cost functions, *k*:

$$\sigma^* = \frac{1}{2-1/k}$$

The elasticity of substitution is inversely related to the cost elasticity *k*. When *k* is very high, σ^* is close to a half, which is the elasticity of substitution of the unconditional matching function *m*. In this case search and recruitment effort are almost exogenous; in the limit both α and γ are equal to one, and the conditional and unconditional matching functions are identical:

$$m = m^* = \frac{uv}{u+v}$$

As *k* decreases towards one, the cost function becomes more linear, and the elasticity of substitution between unemployment and vacancies increases towards one. Intuitively, if marginal costs are approximately constant, workers and firms can adjust their search intensities easily in response to changes in u and v.

(iii) *The elasticity of matching with respect to unemployment is determined by the costs of search relative to the benefits, for workers relative to firms.*

Log-differentiating (24) gives the unemployment elasticity of the unconditional matching function:

$$\eta^* \equiv \frac{\partial \log m^*}{\partial \log u} = \frac{\overline{\eta}^{1-\rho} u^{-\rho}}{\overline{\eta}^{1-\rho} u^{-\rho} + (1-\overline{\eta})^{1-\rho} v^{-\rho}} \quad \text{where} \quad \overline{\eta} \equiv \frac{c_w / \beta}{c_w / \beta + c_f / (1-\beta)}$$

η^* is increasing in $\overline{\eta}$. Hence, the unemployment elasticity is high when the worker's relative cost of search $c_w/(c_w+c_f)$ is high compared with his relative benefit β.

(iv) *The unemployment elasticity η^* of the unconditional matching function is equal to the unemployment elasticity η of the conditional matching function.*

It can be verified directly using (13) and (24) that the unemployment elasticity satisfies:

$$\eta^* = \frac{m}{\alpha u} = \frac{\gamma v}{\alpha u + \gamma v}$$

Hence, by (19), it is equal in equilibrium to the unemployment elasticity η of the conditional matching function (18), which is also the worker's contact probability, or equivalently the proportion of total search effort undertaken by firms. (We show in section 5 below that the equality of the elasticities here is a special case of a more general result.)

(v) *When the cost elasticity k is close to one the unconditional matching function m^* is approximately Cobb-Douglas with constant unemployment elasticity equal to $\overline{\eta}$.*

When k is close to one (the cost function is almost linear) ρ is close to zero, and the elasticity of substitution σ^* between unemployment and vacancies is approximately one. Hence the unconditional matching function (24) is close to its Cobb-Douglas limiting form. From the expression in (iii) for the elasticity we obtain:

$$\eta^* \approx \overline{\eta} \equiv \frac{c_w/\beta}{c_w/\beta + c_f/(1-\beta)}$$

Note however that the function (24) does not have a well-defined limit as $\rho \to 0$ ($k \to 1$) unless $S = c_w/\beta + c_f/(1-\beta)$. With linear costs, this condition is necessary for first-order conditions (13) and (15) to have a solution; otherwise search intensity choices, and hence the rate of matching, would be either zero or infinite.

4.2 Interpreting the Empirical Evidence

These results provide an interpretation of empirical estimates of the aggregate matching function. Blanchard and Diamond (1990) estimated a CES matching function on US data, obtaining an estimate of the elasticity of substitution between unemployment and vacancies of $\hat{\sigma}^* = 0.74$. In the model here, this corresponds to a plausible value for the cost elasticity of k=1.54. Their estimate was not significantly different from 1, corresponding to the Cobb-Douglas limiting case with k=1. Note, however, that our model restricts σ^* to lie in the range 0.5 to 1.0, and it would also be interesting to test the hypothesis $\sigma^* = 0.5$, representing the case of exogenous search effort.

Blanchard and Diamond's corresponding estimate for the unemployment elasticity was 0.48. The interpretation provided by our model is that the costs relative to benefits were slightly lower for workers than for firms, and a slightly higher proportion of total search effort was therefore borne, in equilibrium, by workers. As mentioned above, other studies using the Cobb-Douglas form have obtained an unemployment elasticity between 0.5 and 0.7, suggesting that the costs of search relative to the benefits are somewhat higher for workers – which could result from either a lower surplus share β, or a higher marginal cost of search c_w[5].

Furthermore, by (iv), although empirical studies estimate the unconditional matching function – they do not control for search effort – we can interpret their estimates of the unemployment elasticity as referring to the underlying conditional matching function, which is the value to which the Hosios condition refers.

4.3. Interpreting the Hosios Condition

For the decentralised equilibrium in a matching model to be socially efficient (that is, maximise the social surplus, subject to the matching frictions) the Hosios condition (6) must hold: that is, the elasticity of matching with respect to unemployment must equal the worker's surplus share.

On one level, the efficiency condition is quite intuitive. The decisions made privately by the worker – whether to enter a matching market, how much to invest in search, and whether to accept a particular job offer – all depend on the expected return from search. For an individual worker this depends (see equation (14)) on $(m/u)\beta S$, which is the *average* probability of matching, multiplied by the worker's share of the surplus. But the social benefits of an additional searching worker depend on the *marginal* probability of matching, multiplied by the whole surplus: that is, on $m_u S$. Social efficiency on all three margins obtains when these are equal:

$$\frac{m}{u}\beta = m_u$$

which is the elasticity condition (6).

The unsatisfactory feature of the condition is that the elasticity of matching with respect to unemployment is, in general, endogenous, and we have no intuition as to the conditions under which the elasticity is likely to be more than, less than, or equal to, bargaining power.

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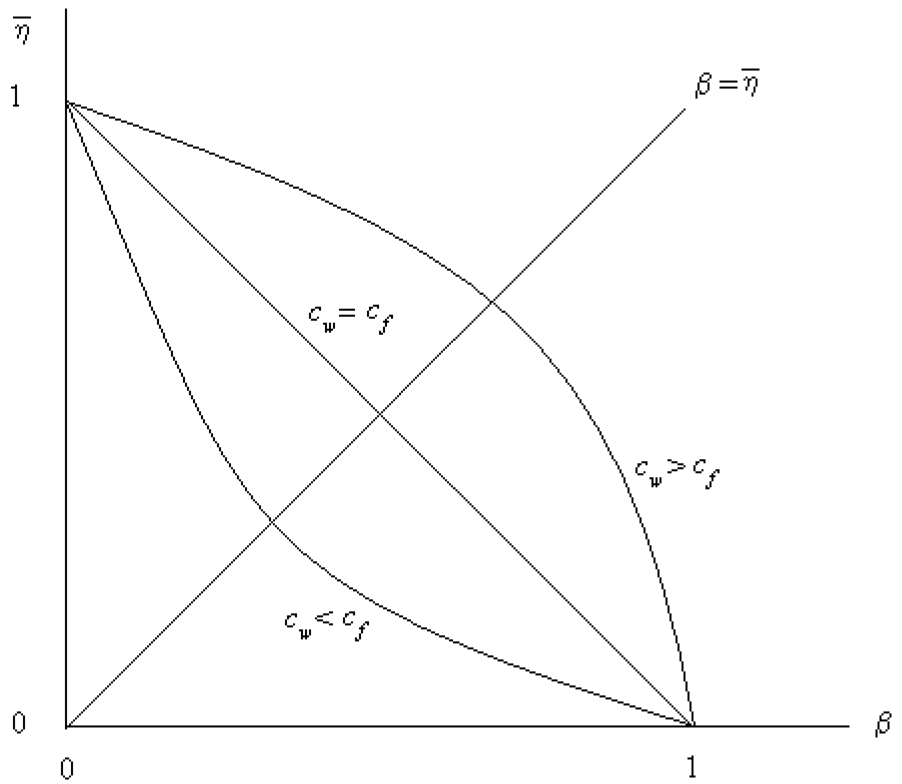
[5] The unemployment elasticity would also be high if workers had a higher discount rate than firms. It is straightforward to generalize the analysis to allow for different discount rates r_w and r_f, giving an

But in the present model, we know exactly what determines the unemployment elasticity. Applying results (iii) and (iv) from section 4.1, the Hosios condition for the matching process of section 3 can be written:

$$\beta = \frac{\overline{\eta}^{1-\rho} u^{-\rho}}{\overline{\eta}^{1-\rho} u^{-\rho} + (1-\overline{\eta})^{1-\rho} v^{-\rho}} \text{ where } \overline{\eta} \equiv \frac{c_w / \beta}{c_w / \beta + c_f /(1-\beta)} \qquad (25)$$

When the cost elasticity k is close to 1, the condition is approximately $\beta = \overline{\eta}$. The crucial insight from this model is that, in equilibrium, the elasticity of matching with respect to unemployment is not an arbitrary quantity, determined separately from relative bargaining power β; instead, it is a decreasing function of β.

Figure 1: The Unemployment Elasticity $\overline{\eta}$ is a Decreasing Function of β.



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unemployment elasticity: $\eta \equiv \frac{c_w (r_w + \lambda) / \beta}{c_w (r_w + \lambda) / \beta + c_f (r_f + \lambda) / (1-\beta)}$

Figure 1 shows $\overline{\eta}$ as a function of β. As β rises from 0 to 1, the unemployment elasticity falls from 1 to 0. There is no tendency towards efficiency: when β is high the unemployment elasticity is low, and vice versa.

There is, however, a unique value of β which achieves efficiency, and this depends on the relative costs c_w/c_f. When the worker's relative costs of search are high, then his relative benefit, β, must also be high if the market is to be efficient. This is quite intuitive. Search by individual workers has two external effects: it raises the probability of matching for firms, and lowers it for other workers. These probabilities depend on the *relative* amounts of search by the two types of agents. There is an efficient relative level, which depends on the relative costs of search: the level that minimises the total cost of achieving any given absolute level of matching. But the search decisions of individual agents depend on benefits as well as costs. Only if the relative benefits happen to be distributed in the same way as the costs will these decisions be efficient.

The Hosios condition determines the particular value of β under which the elasticity takes its optimal value – so that matching is as effective as possible, given the costs of search. When β is higher than its efficient level, workers will invest too much in search (for given relative costs), and firms will invest too little, resulting in a labour market in which the elasticity of matching with respect to unemployment is low. Then there is a low probability of matching for workers and a high probability for firms.

The problem of inefficiency in a matching model can be viewed as a kind of hold-up problem: decisions to enter the unemployment or vacancy pool, and to incur search costs, are investment decisions resulting in a return when a match is formed, but workers and firms are unable to contract to share the costs of these investments appropriately. The Hosios condition identifies the case when private costs happen to be distributed in such a way that a contract is unnecessary.

5. More on the Relationship between the Unconditional and Conditional Matching Functions

The results obtained in section 4 are for a particular matching technology which is of interest because it gives us a CES unconditional matching function. It should be noted, however, that the results for the relationship between the unconditional and conditional matching functions, *m* and *m**, depend on the specification of the search cost functions, and are not specific to one matching technology. In this section we derive some more general results, which hold for any constant elasticity cost functions, and any matching technology with input-augmenting search intensities.

So, assume now that the cost functions for search and recruitment intensity are:

$$C_w(\alpha) = \frac{c_w}{k_w}\alpha^{k_w} \text{ and } C_f(\gamma) = \frac{c_f}{k_f}\gamma^{k_f} \text{ where } k_w, k_f > 1, \text{ and } c_w, c_f > 0 \qquad (26)$$

and that the unconditional matching function is homogeneous of degree 1 in *u* and *v* and satisfies (5).

The *k* parameters are the cost elasticities for the two types of agents, and the *c* parameters capture differences in marginal costs. Equations (13) and (15) for optimal search intensities can then be written:

$$uc_w\alpha^{k_w} = m\beta S \qquad \text{and} \qquad vc_f\beta^{k_f} = m(1-\beta)S \qquad (27)$$

It is convenient to write:

$$\rho_w = 1 - \frac{1}{k_w} \quad \text{and} \quad \rho_f = 1 - \frac{1}{k_f}$$

Then substituting α and γ from (27) into the conditional matching function (5) we obtain an equation for the matching rate *m** with endogenous search:

$$m^* = \tilde{m}\left((\bar{S}m^*)^{1-\rho_w}\,\bar{\eta}^{\rho_w - 1}u^{\rho_w}, (\bar{S}m^*)^{1-\rho_f}(1-\bar{\eta})^{\rho_f - 1}v^{\rho_f}\right) \qquad (28)$$

Equation (28) defines *m** as an implicit function of *u* and *v* - the unconditional matching function. We can then prove the following results for *m**:

PROPOSITION 1: When search intensities are input-augmenting, with constant elasticity cost functions, and the conditional matching function $m(u,v,\alpha,\gamma)$ is increasing and concave in *u* and *v*, and homogeneous of degree 1:

(i) The unconditional matching function m^* is also increasing and concave in u and v, and homogeneous of degree 1.

(ii) The unemployment elasticity η^* of the unconditional matching function is related to the unemployment elasticity η of the conditional matching function by:

$$\eta^* = \frac{\rho_w \eta}{\rho_w \eta + \rho_f (1-\eta)}$$

(iii) When the cost elasticities are equal ($\rho_w = \rho_f = \rho$), the unconditional matching function is given explicitly by: $[m^*(u,v)]^\rho = \bar{S}^{1-\rho} \tilde{m}(\bar{\eta}^{\rho-1} u^\rho, (1-\bar{\eta})^{\rho-1} v^\rho)$

the unemployment elasticities are equal: $\eta^* = \eta$

and the elasticities of substitution are related by: $\frac{1}{\sigma^*} - 1 = \rho\left(\frac{1}{\sigma} - 1\right)$.

PROOF: See Appendix.

The implication of Proposition 1 is that, provided we assume that search and recruitment int ensities are input augmenting with constant cost elasticities, we can effectively ignore the search intensity choice problem and work only with the unconditional matching function m^*, which then has all the standard properties.

Furthermore, if the cost elasticities are equal, we have the result obtained in section 4: the unemployment elasticity of the underlying conditional function is the same as that of the unconditional function, so can be estimated without data on search intensities. Also, the effect of endogenous search intensity is to increase the elasticity of substitution between unemployment and vacancies; and the lower is the cost elasticity k, the more workers and firms will adjust their search in responses to changes in u and v, and hence the higher is the elasticity of substitution between u and v.

In the case when the cost elasticities are not equal, the unemployment elasticity of m^* is affected by their relative values. So if, for example, search costs increase more rapidly for workers than for firms, this will increase the unemployment elasticity. With the particular matching technology in section 3, this would mean that the unemployment elasticity η* would be higher than the equilibrium probability of a successful contact, which would still be equal to η.

5.1 Eliminating Search Intensities from the Matching Model

As suggested above, with constant elasticity cost functions it is possible to remove the search intensity choice from the model and work only with the unconditional matching function. To see this, we can use (26) and (27) to substitute for the cost functions in (14) and (16):

$$rU = z + \frac{m^*(u,v)}{u}\rho_w \beta S \tag{29}$$

$$rV = -x + \frac{m^*(u,v)}{v}\rho_f (1-\beta)S \tag{30}$$

Now if we write: $\beta^* \equiv \dfrac{\rho_w \beta}{\rho_w \beta + \rho_f (1-\beta)}$ and $\bar{\rho} \equiv \rho_w \beta + \rho_f (1-\beta)$

these equations are the standard Bellman equations for the flow value of unemployment and vacancies in a model with exogenous search intensities:

$$rU = z + \frac{\bar{\rho} m^*(u,v)}{u}\beta^* S \tag{31}$$

$$rV = -x + \frac{\bar{\rho} m^*(u,v)}{v}(1-\beta^*)S \tag{32}$$

We know from Proposition 1 that the matching function *m** has all the requisite properties. (It should be remembered, however, that *m** depends on both the cost elasticities and $\bar{\eta}$, the parameter representing the costs relative to benefits for workers – see equation (28).) From (31) and (32) we can see that the effect of a relatively higher search cost elasticity for workers (say) is to increase the share of the surplus that they obtain in equilibrium, equivalent to an increase in bargaining power.

Finally, note that the Hosios efficiency condition (6) for this transformed model is $\beta^*=\eta^*$ which, from Proposition 1(ii), is the same as the condition $\beta=\eta$ for the original model.

References

Acemoglu, D. and Shimer, R. (1999) "Holdups and Efficiency with Search Frictions", *International Economic Review*, 40(4), 827-49.

Blanchard, O. and P. A. Diamond (1990) "The aggregate matching function" in *Growth, Productivity and Unemployment*, ed. P. A. Diamond, Cambridge: MIT Press.

Coles, M. G. and E. Smith (1998) "Turnover externalities with marketplace trading", *International Economic Review*, 40, 851-68.

Cox, D. R. and H. D. Miller (1965) *The Theory of Stochastic Processes*, London: Methuen.

Diamond, P. A. (1982) "Aggregate demand management in search equilibrium", *Journal of Political Economy*, 90, 881-94.

Hall, R. E. (1979) "A Theory of the Natural Unemployment Rate and the Duration of Employment", *Journal of Monetary Economics*, 5, 153-169.

Hosios, A. J. (1990) "On the efficiency of matching and related models of search and unemployment", *Review of Economic Studies*, 57, 279-298.

Moen, E. R. (1997) "Competitive search equilibrium", *Journal of Political Economy*, 105, 385-411.

Mortensen, D. T. and C. A. Pissarides (1999) "New Developments in Models of search in the labor market", *Handbook of Labor Economics*, eds. O. Ashenfelter and D. Card, Amsterdam: North-Holland.

Peters, M. (1984) "Bertrand Equilibrium with Capacity Constraints and Restricted Mobility", *Econometrica*, 52, 1117-1129.

Petrongolo, B. and C. A. Pissarides (2001) "Looking into the Black Box: A survey of the matching function", *Journal of Economic Literature*, 39, 390-431.

Pissarides, C. A (2000) *Equilibrium Unemployment Theory*, 2nd Edition, Cambridge: MIT Press.

Shimer, R. (1996) "Essays in Search Theory". Ph.D. dissertation, Massachusetts Institute of Technology.

Appendix

PROOF of PROPOSITION 1:

(i) Using the homogeneity of $\tilde{m}$, (28) can be written:

$$\frac{m*}{u} = \tilde{m}\left(\left(\frac{\bar{S}m*}{\bar{\eta}u}\right)^{1-\rho_w}, \left(\frac{\bar{S}m*}{\bar{\eta}u}\right)^{1-\rho_f}\left(\frac{v}{u}\right)^{\rho_f}\right)$$

Hence $m*/u$ is an implicit function of v/u, and homogeneity follows immediately.

Now write (28) as: $m*(u,v) = \tilde{m}(X,Y)$ where:

$X = X(m*,u) = (\bar{S}m*)^{1-\rho_w}\,\bar{\eta}^{\rho_w-1}u^{\rho_w}$ and $Y = Y(m*,v) = (\bar{S}m*)^{1-\rho_f}\,(1-\bar{\eta})^{\rho_f-1}v^{\rho_f}$.

$$\Rightarrow \frac{\partial \ln m*}{\partial \ln u} = \frac{\partial \ln \tilde{m}}{\partial \ln X}\left((1-\rho_w)\frac{\partial \ln m*}{\partial \ln u} + \rho_w\right) + \frac{\partial \ln \tilde{m}}{\partial \ln Y}\left((1-\rho_f)\frac{\partial \ln m*}{\partial \ln u}\right)$$

$$\Rightarrow \eta* = \eta\big((1-\rho_w)\eta* + \rho_w\big) + (1-\eta)(1-\rho_f)\eta*$$

$$\Rightarrow \eta* = \frac{\rho_w\eta}{\rho_w\eta + \rho_f(1-\eta)} \qquad (33)$$

Hence $m*$ is an increasing function of u, and similarly of v.

Now differentiate again with respect to ln$u$:

$$\frac{\partial \eta*}{\partial \ln u} = \frac{\rho_w\rho_f}{\big(\rho_w\eta + \rho_f(1-\eta)\big)^2}\left(\frac{\partial \eta}{\partial \ln X}\big((1-\rho_w)\eta* + \rho_w\big) + \frac{\partial \eta}{\partial \ln Y}(1-\rho_f)\eta*\right)$$

Homogeneity of $\tilde{m}$ implies that $\frac{\partial \eta}{\partial \ln X} = -\frac{\partial \eta}{\partial \ln Y}$, so:

$$\frac{\partial \eta*}{\partial \ln u} = \frac{\rho_w\rho_f}{\big(\rho_w\eta + \rho_f(1-\eta)\big)^2}\frac{\partial \eta}{\partial \ln X}\big(\rho_w(1-\eta*) + \rho_f\eta*\big)$$

Also $\frac{\partial \eta*}{\partial \ln u} = \frac{u^2}{m*}m_{11}* + \eta*(1-\eta*)$ and similarly $\frac{\partial \eta}{\partial \ln X} = \frac{X^2}{\tilde{m}}\tilde{m}_{11} + \eta(1-\eta)$.

Substituting both of these expressions in the previous equation:

$$\frac{u^2}{m^*}m_{11}{*} = \frac{\rho_w\rho_f}{\left(\rho_w\eta + \rho_f(1-\eta)\right)^2}\left(\frac{X^2}{\tilde{m}}\tilde{m}_{11} + \eta(1-\eta)\right)\left(\rho_w(1-\eta^*) + \rho_f\eta^*\right) - \eta^*(1-\eta^*)$$

$$= \frac{\rho_w\rho_f}{\left(\rho_w\eta + \rho_f(1-\eta)\right)^2}\frac{X^2}{\tilde{m}}\tilde{m}_{11} + \eta^*(1-\eta^*)\left(\rho_w(1-\eta^*) + \rho_f\eta^* - 1\right)$$

Since *m** is a linear homogeneous function, $m_{11}{*} \le 0$ is necessary and sufficient for concavity. Since $\tilde{m}$ is concave, both terms on the right-hand side are negative and *m** is concave.

(ii) was proved in (33) above.

(iii) Putting $\rho_w = \rho_f = \rho$ in (28) and using the homogeneity of $\tilde{m}$ gives immediately: $[m^*(u,v)]^\rho = \overline{S}^{1-\rho}\tilde{m}(\overline{\eta}^{\rho-1}u^\rho, (1-\overline{\eta})^{\rho-1}v^\rho)$

Putting $\rho_w = \rho_f = \rho$ in (34) gives $\eta^* = \eta$.

The elasticities of substitution are $\sigma^* \equiv \frac{\partial \ln(v/u)}{\partial \ln(m_1{*}/m_2{*})}$ and $\sigma \equiv \frac{\partial \ln(Y/X)}{\partial \ln(\tilde{m}_1/\tilde{m}_2)}$.

Now, $\frac{\eta^*}{1-\eta^*} = \frac{\eta}{1-\eta} \Rightarrow \frac{um_1{*}}{vm_2{*}} = \frac{X\tilde{m}_1}{Y\tilde{m}_2}$ and hence:

$$\ln\left(\frac{m_1{*}}{m_2{*}}\right) - \ln\left(\frac{v}{u}\right) = \ln\left(\frac{\tilde{m}_1}{\tilde{m}_2}\right) - \ln\left(\frac{Y}{X}\right)$$

Differentiating with respect to ln(*v*/*u*):

$$\frac{1}{\sigma^*} - 1 = \left(\frac{1}{\sigma} - 1\right)\frac{\partial \ln(Y/X)}{\partial \ln(v/u)}$$

$$\Rightarrow \qquad \frac{1}{\sigma^*} - 1 = \rho\left(\frac{1}{\sigma} - 1\right) \qquad \square$$