I Am Getting Tired: Effort and Fatigue in Intertemporal Decision-Making

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This Version: August 8, 2007†

Abstract

Evidence on effort-demanding tasks suggests that exerting effort is fatiguing and that overcoming a given threshold of fatigue prevents the agent from providing acceptable performances. This paper proposes a simple dynamic model to study the optimal amount of effort that an agent should exert to manage fatigue over time and possibly avoid exhaustion. Depending on the condition of fatigue of the agent and on the duration of the task to be performed, four testable patterns of optimal effort exertion are suggested.

The model is extended to consider multitasking, showing that this condition induces lower performance profiles and the emergence of thresholds of fatigue beyond which the worker cannot avoid being exhausted. When this is the case a policy intervention, like allowing the worker to take a rest-break, would be advisable

Keywords: Intertemporal choice; Effort; Fatigue; Multitasking, Cognitive load, Rest-breaks.

JEL classification: C60, D90.

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†I wish to thank Angelo Antoci, Francesca Barigozzi, Jordi Brandts, Marco Casari, Matteo Cervellati, Robin Hogarth, Luca Lambertini, Alfredo G. Minerva, Ahmad Naimzada, Arsen Palestini, Raimondello Orsini, Pier Luigi Sacco, Paolo Vanin, A special thank to Erin Krupka for her detailed comments. The usual disclaimer applies.

1 Introduction

Many tasks require the exertion of effort. In general, the amount of effort exerted by a worker is assumed to have two main effects. On one hand, it yields disutility to the worker. On the other, exerting effort positively influences the performance on the assigned task. When the worker internalizes the effects of her effort on the task she is performing (either because she directly enjoys what she produces, or because she receives a compensation), she can determine the optimal amount of effort to exert. In Economics an important stream of research has focused on the situation in which the worker is employed and her effort is not perfectly observable by her principal. In this case the problem for the principal is to find a compensation scheme that gives the worker the incentive to work hard. This paper takes a different approach by focusing on the evidence that exerting effort is fatiguing and that overcoming a given threshold of fatigue prevents the agent from working efficiently. This implies that the optimal choice of the worker does not only depend on the direct effects of effort on utility and performance, but also on the indirect consequences that current effort has on the effort that can be exerted in the future.

The main contribution of this paper is to show that the optimal amount of effort to exert on the job must take into account the accumulation of fatigue and the duration of the task to be performed, and possibly avoid exhaustion. This conclusion is particularly relevant when a worker is employed, because it implies that compensation schemes that require either high or constant level of efforts may be suboptimal. On one hand, high levels of effort determine a rapid accumulation of fatigue, thereby quickly driving the worker to be exhausted and not productive; on the other hand, constant effort neglects the dynamics of accumulation and decumulation of fatigue and their contribution to intertemporal utility maximization. Given the duration of the task to be performed and the fatigue of the worker, I identify four testable classes of optimal effort schedules.

Focusing on tasks requiring high level cognition, the model is useful for studying the effects of multi-tasking. Consistent with the empirical evidence, the model predicts that these conditions determine a reduction in performance as the worker must lower her effort on the assigned job to slow down the accumulation of fatigue. As thresholds of fatigue emerge, this extension also shows that "good" equilibria can be reached only if the agent is sufficiently rested. When this is the case, a policy intervention is desirable. For example, a period of total rest (such as a break, or a holiday) should be imposed to allow the worker to "recharge the battery" and come back to efficiency.

The paper is structured as follows. In next section I discuss the assumption that exerting effort is fatiguing. In Section 3 the benchmark model is presented: this allows to show that it is never optimal to provide a constant level of effort and that four different patterns of effort can be identified as a function of the condition of fatigue and the time-horizon. In Section 4 the model is extended to consider cognitive multitasking and the emergence of thresholds of fatigue. Section 5 considers policy interventions to avoid exhaustion. Section 6 concludes. All proofs are collected in the appendix.

2 Effort and fatigue

The model presented in next section builds on the idea that performing certain tasks requires drawing on a limited stock of resources and that the exhaustion of these resources prevents from being able to provide acceptable performances. In physiology, large evidence shows that muscle contraction involves the depletion of energy[1] and that energy deficiency is a major cause of low performance (Sahlin et al., 1998). This can be interpreted by saying that exerting effort on a physical task is fatiguing and that the accumulation of fatigue beyond a certain threshold impairs performance.

A similar approach has been suggested, for cognitive tasks, with reference to "attention resources" and cognitive fatigue. When speaking of cognitive

[1] Muscle contraction can make use of different sources of energy like, for example, ATP and glycogen, depending on the duration of the task and the force to exert. In the model, this variety of sources is simplified to only one kind of resource.

tasks, the literature focuses on high-level cognition, a broad set that includes mental activities such as thinking, solving problems and exerting self-control, as well as consciously suppressing environmental noise, thoughts and emotions. Low-level cognition is, on the contrary, largely subconscious and it is generally associated to automatic mental activities[2]. A remarkable feature of the dichotomy high/low-level cognition is that the latter is generally considered to be costless, in the sense that i) it requires no significant cognitive effort, ii) it does not induce fatigue (so that it can be potentially borne *ad libitum*) and iii) multiple simultaneous low-level tasks do not significantly interfere one with the other. On the contrary, high-level mental processes are costly. In economics this is typically taken into account by assuming that exerting cognitive effort yields disutility to the decision-maker[3]. An alternative direction is taken by Gabaix et al. (2003)[4]. Their starting point is the evidence that people have finite mental processing speeds, so that paying attention to a specific task prevents from paying attention to another one. According to this perspective, the (shadow) cost of cognition is given by the opportunity cost of the time it takes to perform a certain cognitive task.

Instead of focusing on the assumption that high-level cognitive processes take time, the cost associated to complex decision-making can be approached from a yet different perspective by considering that, after sustained attention, performance on the task degrades becoming more variable and often untimely, with slower reaction times, increases in errors and false responses, lapses in short-term memory and vigilance (Dinges, 1991, 1995, Rosekind, 1995, 1996). The accumulation of fatigue due to complex cognitive activity can affect the performance on both simultaneous and sequential cognitive tasks (Cleeremans et al., 1998; Dorriant et al., 2000; Haines et al., 2001;

[2]This dichotomy parallels the neuroscientific classification between controlled and automatic processing of information within the brain; similar distinctions are also common in social psychology and psychoanalysis and they have been variously labeled. For an overview, see Camerer, Loewenstein and Prelec (2005) and Glimcher (2003); see also Strack and Deutsch (2004) for a psychological model in which social behaviour is the result of the interaction between high- and low-level cognitive processes.

[3]See, for example, Mas-Colell et al. (1995:479).

[4]See also Gifford (2001a, 2001b), De Shazo and Fermo (2004).

French, 2002; Shiv, Fedorikhin, 1999). Interestingly, this result holds even if performance is measured on quite different activities involving, for example, self-regulation, choice-making and problem-solving (Vohs, Faber, 2004; Baumeister et al., 1998, 1999; Muraven, Baumeister, 2000). A possible explanation for this evidence is that these tasks involve high-level cognition and they compete with each other by drawing on the same, limited stock of cognitive resources. As a consequence exerting effort on a cognitively demanding task does not simply consume time, but it literally consumes part of the stock of cognitive resources that are necessary for performing alternative tasks. The experimental finding that cognitive overload induced by a very demanding cognitive task reduces the span of time in which cognitive effort can be successfully exerted (Dorrian et al., 2000; Lorist et al., 2000; Bourne, Yaroush, 2003; Tucker, 2003) is consistent with the *resource depletion hypothesis*.

To the best of my knowledge, this approach has been neglected by the economic literature except for the contributions of Loewenstein and O'Donoghue (2004) and Ozdenoren et al. (2005). Loewenstein and O'Donoghue (2004) consider a limited stock of cognitive resources, labeled as "willpower", which is depleted whenever self-control effort is exerted and that determines the relative weight given to affective and deliberative processes within the brain; Ozdenoren et al. (2005) use the concept of willpower in a dynamic self-control problem, showing that the depletion hypothesis can provide an explanation for the empirical evidence on commitment, intertemporal preference reversals and procrastination. Like a muscle that gets fatigued but builds up strength, they also assume them exercising self-control depletes willpower over the short term but builds it over the longer term. This assumption is inspired by the empirical evidence provided in Muraven et al. (1999) and Muraven and Baumeister (2000), but it is not uncontroversial (Murtagh, Todd, 2004). In my model, instead, the resources to be used to perform a given task are assumed to be renewable, so that there exists an innate tendency to recover from fatigue until a given level is reached. This recovery assumption, that is clearly complementary to the muscle-building assumption of Ozdenoren et al. (2005), is consistent with the evidence on the effect of rest-breaks and sleep on muscular and cognitive fatigue, and it allows to consider recovery from fatigue by referring only to the condition of fatigue of the agent and the effort she exerts.

3 The Model

In this section I study the optimal pattern of effort that an agent, given her condition of fatigue, should devote to an effort-demanding job. I first present the objective function and the transition equation that represents the dynamics of fatigue over time; then I study the optimal control problem both over an infinite and a finite time-horizon.

3.1 The objective function

The worker's preferences are represented by the following intertemporal utility function:

$$V(c(t), e(t)) = \int_{t=0}^{\infty} e^{-\rho t} v(c(t), e(t)) dt \tag{1}$$

where $c(t) \geq 0$ represents instantaneous consumption, $e(t) \geq 0$ is instantaneous effort and $\rho \geq 0$ is a constant discount factor[5].

For simplicity, let the worker be the owner of the unique productive input, the effort $e(t)$ exerted at time t, and the owner of the output y that is obtained according to the production function $h(e(t))$[6]. Assume that the minimum effort requirement is $e(t) > 0$, so that any level of effort above zero is acceptable. No technological progress is allowed for and saving is not possible. As a consequence, consumption at time t depends on production as a function of the effort exerted at time t :

$$c(t) = y(t) = h(e(t)). \tag{2}$$

[5]Assume also that $\partial v(\cdot)/\partial c > 0$ and $\partial v(\cdot)/\partial e < 0$; moreover $\partial^2 v(\cdot)/\partial c^2 < 0$ and $\partial^2 v(\cdot)/\partial e^2 < 0$, with the requirement that $v(\cdot)$ is strictly concave for any admissible pair $(c(t), e(t))$.

[6]Assume also that $y(t) = h(e(t)) > 0$ for $e(t) > 0$, $h(0) = 0$, $\partial h(\cdot)/\partial e > 0$ and $\partial^2 h(\cdot)/\partial e^2 < 0$.

Combining (1) and (2), the intertemporal utility function can be simplified as follows:

$$U(e(t)) = \int_{t=0}^{\infty} e^{-\rho t} u(e(t)) dt \tag{3}$$

where $U(e(t)) = V(c(t), e(t))$ and $u(e(t)) = v(h(e(t)), e(t))$ with $\frac{\partial^2 u(\cdot)}{\partial e^2} < 0$. Normalizing to zero the utility the agent can get from an alternative non effort-demanding task, the function $u(e(t))$, that jointly considers the positive effect of effort exertion on utility (via consumption or compensation) and the disutility of effort, is assumed to be non negative in the relevant range of effort levels. For further reference, let $e^{\max}$ be the level of effort that maximizes the instantaneous utility function, i.e. $e^{\max}$ is such that (omitting the time index): $u_e = \frac{\partial v(\cdot)}{\partial c}\frac{\partial h(e)}{\partial e} + \frac{\partial v(\cdot)}{\partial e} = 0$, where the first term represents the marginal utility of effort via consumption and the second one is the marginal disutility of effort.

3.2 The transition equation

The idea of fatigue rests on the assumption that fatigue can be accumulated or recovered depending on effort exertion. This can be interpreted by saying that effort determines the depletion (or recovery) of a stock of renewable resources that are necessary to perform a given task. To formalize it, a state variable $s(t) \geq 0$ is introduced to indicate the condition of the worker at time t: a large stock of resources means that the worker is *rested* and a low stock condition is equivalent to say that the agent is *fatigued*, when $s(t) = 0$ the agent is defined as *exhausted*.

The condition of the worker changes over time as a function of both the condition of the agent and the effort provided at time t:

$$\dot{s}(t) = g(s(t), e(t)).$$

For concreteness, the following separable form is proposed:

$$\dot{s}(t) = s(t)[\bar{s} - s(t)] - f(e(t)) \tag{4}$$

The first part of the expression is a logistic function, which is commonly used to model the dynamics of renewable resources. This function assumes that the stock of resources is increasing as long as $s \in (0, \bar{s})$, and it implies that the rate of recovery from fatigue is low both when the agent is very fatigued (s is close to zero) and when she is close to the "holiday condition" $\bar{s}$, i.e. the homeostatic condition of non fatigue that the decision-maker reaches if she does not exert any effort. The second part of expression (4) represents the depleting effect of exerting effort on the condition of the worker. This depletion function is positive, strictly increasing and convex for $e(t) > 0$, with $f(0) = 0$. Assume also that the optimal instantaneous effort $e^{\max}$ is high enough to generate fatigue, i.e. $g(s(t), e^{\max}) < 0$ for any $s(t)$[7].

3.3 The optimal control problem

The maximization problem is first formulated over an infinite horizon in order to understand whether it is theoretically possible to carry out an effort-demanding task forever by taking into account the endogenous costs of effort exertion. To better highlight the role of the accumulation of fatigue on the time pattern of effort, no uncertainty exists[8]. Note also that the adoption of a standard intertemporal utility function with exponential discounting at a constant rate implies stationarity for the intertemporal utility function, thus ruling out dynamic inconsistent choices (Strotz, 1955/6).

Formally the intertemporal maximization problem can be formulated as

[7] Fatigue is often defined in terms of a reduced capacity of effort exertion. This suggests that fatigue determines the maximum amount of effort that can be exerted. As this paper focuses on the management of fatigue to avoid exhaustion and not on the maximum amount of effort that can be exerted over time, no upper bound depending on the condition of fatigue is imposed on effort.

[8] See Bénabou and Tirole (2004) for a contribution in which self-regulation depends on imperfect knowledge and imperfect recall.

follows:

$$\max_{\{e(t)\}} \int_{t=0}^{\infty} e^{-\rho t} u(e(t)) dt \tag{5a}$$

$$s.t. \dot{s}(t) = s(t)[\bar{s} - s(t)] - f(e(t)) \tag{5b}$$

$$e(t) \geq 0, \forall t \tag{5c}$$

$$s(t) \geq 0, \forall t \tag{5d}$$

$$s(0) = s_0 > 0 \quad (s_0 \text{ given})$$

where s_0 is the initial condition of fatigue.

Proposition 1 *When the agent is exhausted, no more effort can be exerted*

If the agent is not exhausted, internal solutions can emerge:

Proposition 2 *Given problem (5) and the assumptions on the utility function $u(e(t))$ and the transition function $g(s(t), e(t))$:*

1. *(s^*, e^*) is the unique internal steady state of problem (5);*

2. *(s^*, e^*) has saddle point stability;*

3. *The optimal path is monotonic in the condition of fatigue.*

For expositional convenience the internal steady state, (s^*, e^*) can be used as a reference point:

Definition 1 *An individual is **fatigued** if $s(t) < s^*$ and **rested** if $s(t) > s^*$.*

According to this terminology, the following proposition can be stated.

Proposition 3 *Along the optimal path leading to the steady state:*

1. *the optimal effort is never constant for any discount factor ρ and for any $s(t) > 0$;*

2. *the optimal effort is high and decreasing when the agent is rested and it is low and increasing when she is fatigued.*

The solution to the infinite-horizon problem shows that the worker should take her condition of fatigue into account to maximize her intertemporal utility profile. When she is rested, she should exert more effort than in the steady state. This generates fatigue and monotonically drives the worker towards the steady state. On the contrary, when fatigued the worker should exert lower effort than in the steady state, so that she can recover from fatigue and go towards the steady state solution.

Interestingly, the model shows that, when the endogenous costs of fatigue are taken into account, it can be optimal to provide little effort in order to recover from fatigue. This is a different explanation with respect to the existing literature on efficiency wages or the Principal-Agent literature linking effort choices, incentive schemes and information (Laffont, Martimort, 2001). In particular, the depletion hypothesis is useful to understand that an agent may provide a level of effort that is different from what the Principal desires not because the agent is not appropriately motivated, but because she is trying to manage fatigue over time.

By considering the optimal path leading to the internal steady state, a further implication emerges (see also Ozdenoren et al., 2006). When a worker is fatigued, the optimal path of effort is increasing, implying that also production and consumption levels increase over time. In the context of the revealed-preference theory, this would be interpreted *as if* a fatigued worker had a preference for sequences that improve over time. This conclusion is at odds with discounting models of intertemporal choice in which impatience determines a desire to advance consumption. Nevertheless evidence on preferences for improving profiles (such as increasing consumption paths) has been empirically observed. It has been explained considering utility functions that include savoring or dread of future utilities and negative time preferences (Loewenstein, Prelec, 1991) or reference-dependent utility functions (Loewenstein, 1987). By considering a model with accumulation of fatigue and standard discounting functions, it is possible to provide a complementary explanation that is based on the dynamics of prolonged effort exertion, instead of relying on special utility functions.

3.4 Optimal effort in finite-time horizons

The optimal control problem (5) can be easily applied to finite-time problems. This improves the realism of the model, as in real-life people do not perform tasks forever. Moreover, it makes it possible to provide prescriptions on the optimal pattern of effort to exert as a function of the duration of the task T and of the initial and terminal condition of fatigue, $s(0)$ and $s(T)$[9]. The problem modifies as follows:

$$\max_{\{e(t)\}} \int_{t=0}^{T} e^{-\rho t} u(e(t)) dt \tag{6a}$$

$$s.t. \dot{s}(t) = s(t)[\bar{s} - s(t)] - f(e(t)) \tag{6b}$$

$$e(t) \geq 0, \forall t \tag{6c}$$

$$s(t) \geq 0, \forall t \tag{6d}$$

$$s(0) = s_0 > 0 \tag{6e}$$

$$s(T) = s_T \geq 0 \quad (s_0, s_T, T \text{ given}) \tag{6f}$$

The problem above allows to identify a map of optimal trajectories in the (s, e) space. Given the time-horizon and the initial and terminal conditions of fatigue, it is possible to select a specific path (or part of it) from this map and to make predictions on the optimal path of effort exertion.

Proposition 4 *Given problem (6) the optimal effort to exert is never constant. It is:*

a) *strictly increasing if the worker is fatigued at time* 0 *and at time* T*:* $s(0) < s^*, s(T) < s^*$

b) *strictly decreasing if the the worker is rested at time* 0 *and at time* T*:* $s(0) > s^*, s(T) > s^*$

c) *increasing at time* 0 *and decreasing at time* T *if the worker is fatigued at time* 0 *and rested at time* T*:* $s(0) < s^* < s(T)$

[9] The assessment of the optimal triplet (s_0, s_T, T), which can be interpreted in terms of optimal schedule of work and rest-breaks, is beyond the scope of this paper.

d) *decreasing at time* 0 *and increasing at time* T *if the worker is rested at time* 0 *and fatigued at time* T*:* $s(T) < s^* < s(0)$.

The above propositions provide testable predictions on the optimal path of effort that an agent should exert as a function of the time horizon and the initial and terminal condition of fatigue. Fig. 1 represents some optimal solutions to problem (6) in the $(s(t), e(t))$ space. All the paths are covered in the same time T and they differ only for their initial and terminal conditions. Paths $1, 2$ and 3 (starting at the white circle) correspond to an agent that is fatigued at time 0 (i.e. $s(0) < s^*$), while paths $4, 5$ and 6 (starting with the dot) correspond to an agent that starts working being rested (i.e. $s(0) > s^*$)[10]. Given the initial condition of fatigue and T, the optimal path to choose depends on the trade-off between performance and terminal condition to reach. For example, paths 3, 5 and 6 represent the case of a worker that at time T wants to be in a rested condition, a situation that can occur when an additional effort-demanding tasks is to be performed after the deadline T. In such a case we should observe a decreasing pattern of effort (and performance) at the end of the working period in order to save on the resources to be used afterwards. In other cases an individual may decide to work until exhaustion in order to maximize her performance within the given time-horizon. In this case, the optimal path is the one that hits the vertical axes exactly at time T, i.e. the path starting at $s(0) > 0$ and ending at $s(T) = 0$. As the final part of the path is increasing, the model suggests that the agent should exert increasing effort (a "final rush") as the deadline approaches.

4 Multitasking and thresholds

The model proposed in the previous section illustrates the role of fatigue when a single prolonged task is to be performed. This framework can be extended to environments in which the agent is required to simultaneously perform

[10] For drawing the trajectories, the following functions and parameters have been used: $u(t) = e(t)(100 - e(t))$, $\dot{s}(t) = (5 - s(t))s(t) - e(t)$, $s_0 = 1$ (white circles), $s_0 = 2.73$ (dots), $T = 1.7$.

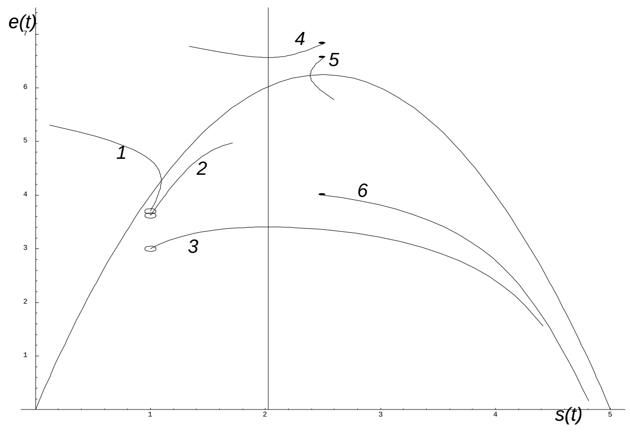

Figure 1: Optimal effort over finite-time horizons.

multiple tasks. This issue has been extensively studied in relation to complex *cognitive* activities, with specific reference to the ability of pilots, drivers and soldiers to maintain sustained attention over time (DeWaard, 1996, Dinges, 1995, St.John et al., 2002). Recent research in Cognitive Psychology has shown that ignoring environmental stimuli (such as noise or stress), as well as ignoring internal stimuli like emotions or thoughts are complex activities that require significant cognitive resources (Wegner, 1989). This suggests that multitasking is a condition that frequently occurs also in every day life (Kirsh, 2000/1, Stone, 2006).

The evidence shows that multitasking negatively affects the performance of people both in terms of the ability to stay concentrated on a task in a specific moment and in terms of the ability to maintain attention over a long time-horizon (Dorrian et al, 2000, Lorist et al., 2000, Smith, Jones, 2002). The negative relation between task demand (also called mental workload[11]) and task performance is particularly evident when there is cognitive overload (Shiv, Fedorikhin, 1999). For example, there is evidence that cognitive overload interferes with self-regulating behaviour, as it is shown by those

[11] For a review of the literature on mental workload, see Gopher and Donchin (1986) and O'Donnell and Eggmeier (1986). For research on mental workload and mental overload (called also the workload redline) see De Waard (1996) and references therein.

people that deviate from a diet or a well-intentioned saving program when they are experiencing stress (Bourne, Yaroush, 2003; Shiffman, Waters, 2004; Herman, Polivy, 2003) or they are cognitively fatigued (Vohs, Faber, 2004; Muraven et al., 1998; Baumeister et al., 1998; Vohs, Heatherton, 2000).

The Cognitive literature explains the relationship between multitasking and performance by arguing that complex tasks all draw on the same stock of cognitive resources (Baumeister et al., 1995). This implies that the agent should reduce the effort exerted on her original task in order to avoid a too rapid depletion of the available stock of resources. As it turns out, reducing the level of effort may not be enough, as multitasking determines the emergence of thresholds of fatigue beyond which the agent, thought optimally choosing her effort, cannot avoid being exhausted.

To introduce the role of multitasking in the model, consider a noisy environment that is to be ignored while the agent is performing her original task. The exogenous stimulus is assumed to have no effect on the utility function, but it has an impact on the cognitive condition of the agent. This can be modeled by assuming that the environment constantly drains an amount $k > 0$ of cognitive resources, so that overall cognitive depletion is given by the sum of the efforts the individual exerts to perform her original task and to ignore the exogenous source of cognitive load. The cognitive load condition is assumed to be permanent, so that the following modified version of the state equation (4) can be considered:

$$\dot{s}(t) = s(t)[\bar{s} - s(t)] - f(e(t) + k) \quad \text{for any } t \tag{7}$$

Considering a cognitively depleting environment, the proposition below is consistent with the empirical observation that multitasking negatively impacts on the performance on effort-demanding tasks.

Proposition 5 *For any condition of fatigue, the optimal effort $e(t)$ to exert is non increasing in the environmental cognitive demands k.*

According to the level of cognitive load, it is possible to distinguish between environments with cognitive overload from environments with mild cognitive load.

Definition 2 *An environment is cognitively depleting when k is such that $\dot{s}(t) = g(s(t), e(t), k) < 0$ for any $e(t) \geq 0, s(t) \geq 0$.*

An environment is overloading when, regardless of how much is exerted, the agents gets more and more fatigued, eventually until exhaustion. This implies that no internal steady state can be reached and that, over a long time-horizon, the only point satisfying the transversality condition is the corner solution $(0, 0)$. According to Proposition 1, this implies that no more effort can be exerted.

In case of a mild cognitive load, there exist conditions of fatigue such that the individual can recover if she exerts little effort. This makes it possible (but not necessary) that an internal steady state (s^{**}, e^{**}) exists.

Proposition 6 *With multitasking:*

1. *If $k > e^*$, no internal steady state exists.*

2. *If $k \in (0, e^*)$, a unique internal steady state (s^{**}, e^{**}) with saddle point stability exists, with $s^{**} = s^*$ and $e^{**} = e^* - k$.*

3. *If $k \in (0, e^*)$, there exists a threshold level of fatigue $s_c \in (0, s^{**}]$ such that, if $s_0 < s_c$, the unique solution is $(0, 0)$.*

The existence of a threshold of fatigue means that it is possible to reach the internal steady state only when the agent is sufficiently rested. On the contrary, when the worker is beyond that threshold, the agent is out of the basin of attraction of the internal solution. When this is the case, lowering the effort on the task to be performed is not enough to avoid the exhaustion of the worker over a long time-horizon. This holds even if the agent exerts no effort on the original task.

5 Local equilibria and rest-breaks

In the previous sections two kinds of steady states emerge: the corner solution $(0, 0)$ and the internal steady states (s^*, e^*) and (s^*, e^{**}). The former is a "bad" equilibrium, since it is associated to a zero utility profile. The latter are a "good" equilibria, as they are associated to positive consumption levels over the infinite time horizon. Therefore it would be desirable to reach these internal points and avoid the dominated corner solution $(0, 0)$.

The previous analysis has shown that reaching the internal steady state is always possible in the benchmark case, provided that the agent is not exhausted. Nevertheless, in case of mild cognitive load the internal steady state (when it exists) cannot be always reached because there exist conditions of fatigue beyond which the agent optimally chooses the level of effort to exert and yet she is driven toward the undesirable steady state in which she is exhausted and cannot exert any more effort. Note that this behavior is neither due to a lapse in rationality, nor to the intertemporal utility function of the agent: the suboptimal outcome is due to the fact that it is impossible to recover, even when no effort is exerted on the original task. When this occurs, two solutions are possible to avoid the exhaustion of the agent. The first one is to eliminate the source of multitasking or to reduce its impact on the agent. This calls for the design of ergonomic environments that improve the productivity and safety of people, a need that is recognized both for physical and cognitive tasks. The second solution requires to the worker to leave the depleting environment. In other words, to avoid exhaustion, it can be better to take a rest-break, or a holiday, until the worker has recovered "enough" resources to come back to efficiency[12]. From this perspective, rest-breaks (or holidays) are important not because people like them (which, after all, seems to be true true for many people), but because they allow the worker to avoid exhaustion, recover from fatigue and to be productive in the long run.

[12] In the case of mild multitasking, this means that the worker should rest until she reaches the threshold of fatigue s_c and she is within the basin of attraction of the good equilibrium. In the case of overload, in which there exists no internal steady state to approach, what "enough resources" (or, equivalently, "enough time of rest") means is still a matter of investigation (Bechtold et al., 1984, Bechtold, Thompson, 1993; Garcia Sanchez, Vazquez Mendez, 2005; Tucker, 1999, 2001, 2003).

6 Conclusions

In the economic literature, effort is generally taken into account by assuming that it is a productive input, but it yields disutility to the decision-maker. In a short time horizon this is a plausible way to take effort into account. Nevertheless, in many situations people must exert effort for prolonged periods of time and the empirical evidence shows that this seriously impacts on performance: students get distracted, workers lose concentration and efficiency, drivers and pilots incur in a higher probability of incurring in an accident. Similar degrading patterns in performance when prolonged effort is exerted are common in many activities people do. Moreover, the accumulation of fatigue gets faster when multiple effort-demanding tasks are to be performed at the same time, as it is the case of people trying to ignore stress or noise in a workplace.

This paper introduces fatigue in a dynamic model of intertemporal decision-making. The critical assumption is that the worker has a certain amount of resources that are depleted or recovered depending on effort exertion. The model shows that it is not optimal to provide constant effort, as this would neglect the endogenous cost of effort on the dynamics of fatigue. More specifically, the optimal pattern of effort depends on the duration of the task as well as on the initial and terminal condition of fatigue. By considering finite-time horizons, four testable patterns of optimal effort exertion are identified.

The model is extended to consider multitasking, showing that this induces lower performance profiles and the emergence of thresholds of fatigue beyond which the worker cannot avoid being exhausted. In these cases modifying the environment according to ergonomic principles, or allowing the worker to take a break and leave the workplace until she has rested enough, would be advisable.

7 Appendix

7.1 Proof of proposition 1

When the agent is exhausted at time τ, $\dot{s}(\tau) = g(0, e(\tau)) = -f(e(\tau))$ must be nonnegative in order to prevent the state variable to become negative. Since $f(e(\cdot))$ is non negative, the only admissible continuation path is $e(t) = 0$ for all $t \geq \tau$.

7.2 Proof of proposition 2

I now focus on internal solutions disregarding corner solutions. First consider the current-value Hamiltonian is (the time index and the arguments are omitted when no confusion arises):

$$H(e, s, m) = u + m[s(\bar{s} - s) - f] \tag{8}$$

The first order condition is, for an internal solution:

$$H_e = u_e - mf_e = 0 \tag{9}$$

which implies

$$m = \frac{u_e}{f_e} > 0 \tag{10}$$

where the sign of the costate variable is given by u_e and f_e being positive. Note that the current-value Hamiltonian function is jointly concave in e and s, meaning that the first order conditions are not only necessary, but also sufficient to guarantee that the solution is optimal, provided that the limiting transversality condition, $\lim_{t\to\infty} m(s - s^*) = 0$, is satisfied. This is indeed true because the utility function is strictly concave in e, the motion of the state variable is jointly concave in e and s and the shadow price of the cognitive resource m is positive for non satiating consumption levels. The maximum principle yields:

$$\dot{s} = s(\bar{s} - s) - f \tag{11a}$$

$$\dot{m} = m(\rho + 2s - \bar{s}). \tag{11b}$$

Claim 1

By differentiating the first order condition (9) and substituting (10) and (11b), it is possible to express the dynamics of optimal effort as a function of the state and the control variables:

$$\dot{e} = \frac{f_e}{u_{ee} - m f_{ee}} \dot{m} \tag{12}$$

$$= \frac{u_e f_e}{u_{ee} f_e - u_e f_{ee}} (\rho + 2s - \bar{s}). \tag{13}$$

which implies that the optimal effort paths are given by the solution of the following dynamic system:

$$\dot{s} = s(\bar{s} - s) - f \tag{14a}$$

$$\dot{e} = \frac{u_e f_e}{u_{ee} f_e - u_e f_{ee}} (\rho + 2s - \bar{s}) \tag{14b}$$

The internal steady state (s^*, e^*) solves system (14) with equality. Since the depletion function $f(e)$ is a strictly monotonic function and $e^{\max}$ is not feasible because it implies $\dot{s} < 0$ by assumption, e^* is unique. Thus there exists only one steady state:

$$s^* = \frac{\bar{s} - \rho}{2}$$

$$e^* = f^{-1}(s^*(\bar{s} - s^*))$$

Where f^{-1} indicates the inverse of the depletion function.

Claim 2

To check the local stability properties of the system around (s^*, e^*), consider the following Jacobian:

$$J = \begin{bmatrix} \frac{\partial \dot{s}}{\partial s} & \frac{\partial \dot{s}}{\partial e} \\ \frac{\partial \dot{e}}{\partial s} & \frac{\partial \dot{e}}{\partial e} \end{bmatrix} \tag{15}$$

$$= \begin{bmatrix} \bar{s} - 2s & -f_e \\ 2\frac{u_e f_e}{u_{ee} f_e - u_e f_{ee}} & \frac{u_{ee} f_e + u_e f_{ee}}{u_{ee} f_e - u_e f_{ee}} (\rho + 2s - \bar{s}) \end{bmatrix} \tag{16}$$

In (s^*, e^*), $\det(J) = 2\frac{u_e f_e^2}{u_{ee} f_e - u_e f_{ee}} < 0$. Since the determinant of a matrix coincides with the product of its eigenvalues, this implies that the eigenvalues are real and with opposite signs, so that the steady state has saddle point stability.

Claim 3

See Kamien and Schwartz (1991:179)

7.3 Proof of proposition 3

By considering the transition equation, for any value of s there exists a (unique) level of effort, call it e_s, such that $\dot{s} = 0$. Kamien and Schwartz (1991:179-180) show that, along the stable manifold, for $s < s^*$ ($s > s^*$) then the optimal value of effort is such that $e < e_s$ ($e > e_s$), a condition that can be interpreted saying that a fatigued (rested) agent exerts low (high) effort along the optimal path.

Consider (14b). Given the assumptions on the utility and the depletion function, the ratio $\frac{u_e f_e}{u_{ee} f_e - u_e f_{ee}}$ is always negative, so that $sign(\dot{e}) = sign(\bar{s} - \rho - 2s) = sign(s^* - s)$. This implies that, out of the steady state, i) the optimal level of effort changes continuously over time even if $\rho = 0$, and ii) the level of effort is increasing (resp. decreasing) iff the agent is fatigued (rested).

7.4 Proof of proposition 4

Consider the map of trajectories that solve problem (5). Every trajectory is optimal because it satisfies the foc (9) and the canonic equations (11), but it solves a different problem, as characterized by specific initial and terminal conditions, and a given terminal time T. Consider the trajectories that start from a generic s_0 and end in s_T. As trajectories can never intersect, there exists only one trajectory that is covered in the time-horizon T.

7.5 Proof of proposition 5

The Hamiltonian function corresponding to a multitasking environment is (omitting the time index and not considering the non negativity constraints):

$$H(s, e, m, k) = u(e) + m[s(\bar{s} - s) - f(e + k)] \tag{17}$$

and the corresponding foc (for internal solutions) is

$$H_e(s, \hat{e}, m, k) = u_e(\hat{e}) - m f_e(\hat{e} + k) = 0 \tag{18}$$

where $\hat{e}$ is the optimal level of effort and m is strictly positive. Since the Hamiltonian function is strictly concave and the depleting function is convex, applying the implicit function theorem to the foc (18) yields:

$$\frac{d\hat{e}}{dk} = \frac{m\tilde{f}_{ee}(\hat{e}+k)}{H_{ee}(s,\hat{e},m,k)} \leq 0 \tag{19}$$

where the equality holds only if the depletion function is linear. Note: the above relationship between optimal effort and environmental load holds everything else equal, with specific reference to the value of $m(t)$.

7.6 Proof of proposition 6

With mild cognitive load, the new internal steady state must solve the following system with equality

$$\dot{s} = s(\bar{s}-s) - f(\hat{e}+k) \tag{20a}$$

$$\dot{e} = \frac{u_e(\hat{e})f_e(\hat{e}+k)}{u_{ee}(\hat{e})f_e(\hat{e}+k) - u_e(\hat{e})f_{ee}(\hat{e}+k)}(\rho+2s-\bar{s}) \tag{20b}$$

Clearly, the steady state value of s is the same as the benchmark case: $s^{**} = s^* = \frac{\bar{s}-\rho}{2}$, while the steady state level of effort decreases with $k : e^{**} = f^{-1}(s^*(\bar{s}-s^*)) - k = e^* - k$. When $e^{**} > 0$, uniqueness and stability of the new internal steady state can be assessed as in the benchmark case. To give an intuition of why a threshold $s_c \in (0, s^*)$ emerges, consider that, when $k < e^*$, the $\dot{s} = 0$ locus shifts downward and intersects the e axis in two points, s_1 and s_2, such that: $0 < s_1 < s^* < s_2 < \bar{s}$. For $s \leq s_1$, $\dot{s}$ is negative and the system is lead westward. This has two implications. First: the internal steady state cannot be reached. Second, at a finite time T, the system hits the vertical axis and no more effort can be exerted. Clearly, this condition ensures that the integral of the intertemporal utility function converges.

References

[1] Ariely, D. and Ziv Carmon (2003). Preferences Over Sequences of Outcomes, in Time and Decision: Economic and Psychological Perspectives on Intertemporal Choice. George Loewenstein, Daniel Read, and Roy Baumeister eds., Russell Sage Foundation Press.

[2] Baumeister, Roy E., Ellen Bratslavsky, Mark Muraven, and Dianne M. Tice (1998). Ego Depletion: Is the Active Self a Limited Resource? Journal of Personality and Social Psychology, 74(5), 1252-1265.

[3] Bénabou, Roland and Jean Tirole (2004). Willpower and Personal Rules. Journal of Political Economy, 112, 848–886.

[4] Bills, A. G. (1937). Fatigue in Mental Work. Physiological Review, 17, 436–453.

[5] Bonnardel, Nathalie and Annie Piolat (2003). Design Activities: How to Analyze Cognitive Effort Associated to Cognitive Treatments? The International Journal of Cognitive Technology, 8(1), 6-15.

[6] Bonnefond, A., P.Tassi, J.Roge, A.Muzet (2004). A Critical Review of Techniques Aiming at Enhancing and Sustaining Worker's Alertness during the Night Shift. Industrial Health, 42, 1-14.

[7] Bourne, Lyle E. Jr. and Rita A. Yaroush (2003). Stress and Cognition: A Cognitive Psychological Perspective. NASA/CR, 212282.

[8] Camerer, C., Loewenstein G. and Prelec D. (2005). Neuroeconomics: How Neuroscience can Inform Economics. Journal of Economic Literature, XLIII, 9-64.

[9] Cleeremans A., Destrebecqz A. and Boyer M. (1998). Implicit Learning: News from the Front. Trends in Cognitive Science, 2(10), 406-416.

[10] De Waard, Dick (1996). The Measurement of Drivers' Mental Workload. Ph.D. thesis,University of Groningen. Haren, The Netherlands: University of Groningen, Traffic Research Centre.

[11] De Shazo, J.R. and German Fermo (2004). Implications of Rationally-Adaptive Pre-choice Behavior for the Design and Estimation of Choice Models, School of Public Policy and Social Research, UCLA, working paper.

[12] Dinges, D.F. (1995). Performance Effects of Fatigue. NTSB/NASA Fatigue Symposium, 41-46.

[13] Dorrian, Jill, Nicole Lamond and Drew Dawsonf (2000). The Ability to Self-Monitor Performance when Fatigued. Journal of Sleep Research, 9, 137-144.

[14] Frederick, S., Loewenstein, G. and O'Donoghue, T. (2002). Time Discounting and Time Preference A Critical Review. Journal of Economic Literature, XL, 351-401.

[15] French, Jon (2002). A Model to Predict Fatigue Degraded Performance. Proceedings of the 2002 IEEE 7th Conference on Human Factors and Power Plants, 4-9.

[16] Gabaix, Xavier, David Laibson, Guillermo Moloche and Stephen Weinberg (2003). The Allocation of Attention: Theory and Evidence, Working paper, MIT.

[17] Gifford, Sharon (2001a). Limited Attention as a Bound on Rationality, mimeo.

[18] Gifford, Sharon (2001b). Endogenous Information Costs, mimeo.

[19] Glimcher, Paul W. (2003) Decisions, Uncertainty and the Brain: The Science of Neuroeconomics, MIT Press.

[20] Gopher, Daniel and Emanuel Donchin (1986). Workload: an Examination of the Concept. In Boff, K., Kaufman, L., & Thomas, J. (Eds.), Handbook of Perception and Human Performance, Vol. II. New York: Wiley.

[21] Kahneman, D. (1973). Attention and effort. Englewood Cliffs, N J: Prentice-Hall

[22] Kanfer, R., P.L. Ackerman, T.C. Murtha, B. Dugdale, L. Nelson (1994). Goal Setting, Conditions of Practice, and Task Performance: A resource allocation perspective. Journal of Applied Psychology, 79, 826-835.

[23] Haines M.M., Stansfeld S. A., Job R. F. S., Berglund B. and J. Head (2001). Chronic Aircraft Noise Exposure, Stress Responses, Mental Health and Cognitive Performance in School Children. Psychological Medicine, 31, 265-277.

[24] Herman, C. Peter and Janet Polivy (2003). Dieting as an Exercise in Behavioral Economics, in George Loewenstein, Daniel Read and Roy F. Baumeister, eds., Time and Decision: Economic and Psychological Perspectives on Intertemporal Choice. New York: Russell Sage Foundation, 459-489.

[25] Hockey, G. Robert J. (1997). Compensatory Control in the Regulation of Human Perfomance under Stress and High Workload: A Cognitive-Energetical Framework. Biological Psychology, 45, 73-93.

[26] Kirsh, D. (2000/1). A Few Thoughts on Cognitive Overload, Intellectica, 19-51.

[27] Laffont, J.J.; Martimort D. (2001). The Theory of Incentives: The Principal-Agent Model. Princeton University Press

[28] Loewenstein, George (1987). Anticipation and the Valuation of Delayed Consumption. Economic Journal, 97, 666- 84.

[29] Loewenstein, George and Drazen Prelec (1991). Negative Time Preference. American Economic Review, 81, 347-52.

[30] Loewenstein, George and Nachum Sicherman (1991). Do Workers Prefer Increasing Wage Profiles? Journal of Labor Economics, 9(1), 67-84.

[31] Lorist, Monicque M., Merel Klein, Sander Nieuwnhuis, Ritske De Jong, Gijsbertus Mulder and Theo F.Meijman (2000). Mental Fatigue and Task Control: Planning and Preparation. Psychophysiology, 37, 614–625.

[32] Mas-Colell, Andreu, Michael Whinston, and Jerry R. Green (1995). Microeconomic Theory. Oxford University Press.

[33] Muraven, Mark R. (1998). Mechanisms of Self-Control Failure: Motivation and Limited Resource. Ph.D. Dissertation, Case Western Reserve University.

[34] Muraven, Mark R. and Roy F. Baumeister (2000). Self-Regulation and Depletion of Limited Resources: Does Self-Control Resemble a Muscle? Psychological Bulletin, 126, 247-259.

[35] Muraven, Mark; Baumeister, Roy and Tice, Dianne (1999). Longitudinal Improvement of Self-Regulation Through Practice: Building Self-Control Strength Through Repeated Exercise. Journal of Social Psychology, 139(4), pp. 446-457.

[36] Muraven, Mark, Dianne M. Tice and Roy F. Baumeister (1998). Self-Control as Limited Resource: Regulatory Depletion Patterns. Journal of Personality and Social Psychology, 74, 774-789.

[37] Murtagh, Anne M. and Susan A. Todd (2004). "Self-Regulation: A Challenge to the Strength Model." Journal of Articles in Support of the Null Hypothesis, 3(1), 19-51.

[38] Norman, D., & Bobrow, D. (1975). On data-limited and resource-limited processes. Cognitive Psychology, 7, 44-64

[39] O'Donnell, Robert D. and Thomas F.Eggemeier (1986). Workload Assessment Methodology. In Boff, K., Kaufman, L., &Thomas, J. (Eds.), Handbook of Perception and Human Performance, Vol. II. New York: Wiley.

[40] Ozdenoren, Emre, Stephen Salant and Dan Silverman (2005). Willpower and the Optimal Control of Visceral Urges, mimeo.

[41] Rogers, N.K., J.Dorrian, D.F. Dinges (2003). Sleep, Waking and Neurobehavioral Performance. Frontiers in Bioscience, 8, s1056-1067.

[42] Schmeichel, Brandon J., Kathleen D. Vohs and Roy F. Baumeister (2003). Intellectual Performance and Ego Depletion: Role of the Self in Logical Reasoning and Other Information Processing. Journal of Personality and Social Psychology, 85, 33-46.

[43] Sahlin K., M. Tonkonogi, K. Söderlund (1998). Energy Supply and Muscle Fatigue in Humans, Acta Physiologica Scandinavica, 162 (3), 261–266.

[44] Shiffman, Saul and Andrew J. Waters (2004). Negative Affect and Smoking Lapses: A Prospective Analysis. Journal of Consulting and Clinical Psychology, 72(2), 192-201.

[45] Shiv, Baba and Alexander Fedorikhin (1999). Heart and Mind in Conflict: The Interplay of Affect and Cognition in Consumer Decision Making. Journal of Consumer Research, 26, 278-292.

[46] Smith, A. P., Jones, D. M. (1992). Noise and Performance. In Handbook of Human Performance, Vol. 1 (ed. D. M. Jones and A. P. Smith), Academic Press: London, 1-28.

[47] St.John, Mark, David A.Kobus and Jeffery G.Morrison (2002). A Multi-Tasking Environment for Manipulating and Measuring Neural Correlates of Cognitive Workload. Proceedings of the 2002 IEEE 7th Conference on Human Factors and Power Plants. New York, NY:IEEE, 7.10 - 7.14.

[48] Strack, Fritz and Roland Deutsch (2004). Reflective and Impulsive Determinants of Social Behavior. Personality and Social Psychology Review, 8(3), 220-247.

[49] Stone, L. (2006). Attention: The Real Aphrodisiac. Speech to O'Reilly Emerging Technologies Conference, March 7. Available online: http://conferences.oreillynet.com/cs/et2006/view/e_sess/8290

[50] Strotz, R. H. (1955-1956). Myopia and Inconsistency in Dynamic Utililty Maximization. Review of Economic Studies, 23(3), 165-80.

[51] Tucker, Philip (2003). The Impact of Rest Breaks upon Accident Risk, Fatigue and Performance: a Review. Work & Stress, April/June, 17(2), 123 -137.

[52] Vohs, Kathleen D. and Ronald L. Faber (2004). Spent Resources: Self-Regulation and Impulse Buying. mimeo.

[53] Vohs, Kathleen D. and Todd F. Heatherton (2000). Self-Regulatory Failure: A Resource-Depletion Approach. Psychological Science, 11, 249-254.

[54] Wegner, Daniel M. (1989). White Bears and Other Unwanted Thoughts: Suppression, Obsession, and the Psychology of Mental Control. New York: Viking/Penguin.

[55] Wirl, Franz (2004), Thresholds in Concave Renewable Resource Models, Ecological Economics, 48, 259–267.