

# Labor Market Friction, Firm Heterogeneity, and Aggregate Employment and Productivity\*

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## Abstract

The paper is based on a synthesis of a "product variety" version of the firm life cycle model developed by Klette and Kortum (2004) and an equilibrium search model of the labor market with job to job flows introduced by Mortensen (2003). In the construction, a continuum of intermediate product and service varieties are produced with labor that serve as inputs in the production of a final good. Intermediate goods producers generally differ with respect to their productivity. New firms enter and continuing firms grow by developing new product varieties. The time required to match workers and jobs in the model depends on the total search effort of workers and the total number of vacancies. Workers can search both while employed and unemployed. Wages are set continuously as the outcome of a bargaining problem over current output. A job separation occurs if either a worker quits or a job is destroyed. We show that a general equilibrium solution to the model exists and that the equilibrium is broadly consistent with observed dispersion in firm productivity, wages, and the relationship between them as well as patterns of worker flows. The model implies that frictions, both in the labor market and in the firm growth process, can be important determinants of aggregate productivity as well as aggregate employment.

## 1 Introduction

Firm productivity differentials are large and persistent. Average wages paid by firms are positively correlated with firm productivity and more productive firms are larger and are more likely to export. Empirical evidence supports the view that workers move from lower to higher paying jobs. These differences imply that the reallocation process may be an

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important determinate of aggregate productivity as well as employment. The purpose of this paper is to develop a tractable equilibrium model that explains these and other stylized facts relating worker flows, wages, and productivity across firms. The equilibrium solution to the model also provides a framework for studying the determinants of the distribution of productivity across firms as well as the level of aggregate employment and productivity when frictions in the labor market are present.<sup>1</sup>

The paper is based on a synthesis of a "product variety" version of the firm life cycle model developed by Klette and Kortum (2004) and an equilibrium search model of the labor market in the spirit of that introduced in Mortensen (2003). Household value future streams of consumption and leisure. A continuum of intermediate product and service varieties are produced with labor input. These serve as inputs in the production of a final good which can be used either for consumption or investment.

Both potential and continuing firms invest in costly R&D. A given measure of potential firms exist. Those that are successful enter the economy as operating firms. Continuing firms grow by creating and developing new product lines. The creator of a new variety has a cost advantage in supplying the product which generates the surplus needed to motivate the investment required to innovate. The firm's derived demand for labor is limited by the demand for the firm's portfolio of products. Existing products are destroyed at an exogenous rate and a firm dies when all of its products lines are destroyed.

Time is required to match workers and jobs in the model. Workers, who are identical by assumption, search at endogenous intensities both while employed and unemployed. Wages are set as the outcome of a strategic bargaining problem over match surplus. A job separation occurs if either a worker quits or a job is destroyed. Firms with vacancies also invest in recruiting workers. The rates at which workers and vacancies meet are determined by a matching function that depends on aggregate worker search and recruiting effort.

In general steady state equilibrium, consumption and leisure, the total number of product lines supplied by each firm type, the number of worker employed by each firm type, and the number of employed workers are all stationary. The steady state equilibrium distributions of both labor productivity and wages across firms and the employment rate reflect these steady state conditions as well as optimality requirements. A steady state equilibrium is defined as a consumption flow, a firm entry rate, a product creation rate and a measure of products supplied for each firm type, a firm recruiting strategy, and a worker search strategy that satisfy optimality and the steady state conditions. The existence of at least one steady state equilibrium solution is established. The equilibrium solution illustrates the importance of both labor market frictions and the cost of firm growth as determinants of aggregate productivity and employment.

## 2 Danish Wage and Productivity Dispersion

Both productivity and wage dispersion are large and persistent in the micro data. (Early work documenting productivity dispersion is reviewed in Bartelsman and Doms (2000). See Davis and Haltiwanger (1991) on wage dispersion.) There is reduced form evidence that workers are reallocated from less to more productive firms as well. (Foster, Haltiwanger,

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<sup>1</sup>In this respect, the paper is closely related to Lagos (2006).

and Krizan (2001).) Finally, Lentz and Mortensen (2006) estimate a structural model of firm dynamics, closely related to that studied here, using Danish longitudinal firm data. It implies that about half of productivity growth can be attributed to reallocation within firm birth cohort as reflected in the higher relative growth rates of more productive firms.

Nagypal (2004) finds that over half of all U.S. prime age full time workers who separate in a month are reemployed with another employer in the next month and that 70% of those who don't leave the labor force experience such a job-to-job movement. That workers who quit to move to another employer typically receive an increase in earnings has long been known. (Bartel and Borjas (1981) and Mincer (1986) represent early work on the finding.) Recently structural models of job to job movements have added more details. For example, Jolivet, Postel-Vinay and Robin (2004) show that an estimated off-the-shelf search on-the-job model does a good job of explaining the observed extent to which the distribution of wages offered to new employees is stochastically dominated by the distribution of wages earned in 9 out of 11 OECD countries. Christensen, et al. (2005) estimate a structural model that allows for an endogenous choice of search intensity using Danish matched worker-employer data. All of these studies supports the basic view that wage dispersion exists in the sense that different firms pay similar workers differently and that workers respond to these differences by moving from lower to higher paying employers.

This paper is based on evidence that wage dispersion is induced by productive firm heterogeneity through "rent sharing." Given search friction, match rents are larger at more productive firms. Hence, if the wage is determined by some form of rent sharing, then we should see a positive cross section relationship between the average wage paid and firm productivity.

In this section, we present Danish evidence for wage and productivity dispersion as well as a positive association between them drawn from an longitudinal files on the value added (Y), full time equivalent (FTE) employment (N) and the wage bill (W) paid by privately owned Danish firms. These firm accounting data are collected annually in a survey conducted by Statistics Denmark and are supplemented from tax records. The survey is a rolling panel and the sampling of firms is based on firm size and revenue.<sup>2</sup>

In Figure 1, cross firm probability density functions for wage cost per worker, as reflected in the ratio of the wage bill to FTE employment ( $W/N$ ), and for average productivity per worker, as measured by the ratio of valued added to employment ( $Y/N$ ), are plotted for the year 2002.<sup>3</sup>The dashed lines represent a smoothed version of the raw data. The solid lines are log normal densities with the same mean and variance as the data, drawn for comparison. The first point to notice is that dispersion in both the average wage paid and labor productivity is large. Specifically, four fold productivity differentials are well within the 5-95 percentile support of the data as well as over 100% differences in the average wages paid. Second, although the right tails of both empirical distributions are slightly fatter than implied by the log normal, the log of both variable can be well approximated by normal distributions. The relative positions of the two distributions suggest that average labor

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<sup>2</sup>Only firms with 5 or more employees and revenue above 500M DKK are included. All firms with labor force at or above 50 are included in the survey. The sampling proportions for 20-49 workers is 50%, for 10-19 is 20%, and for 5-9 is 10%.

<sup>3</sup>At the moment we have data for 1999-2002 inclusive. All the properties characteristics of the data pointed out here are virtually identical for each of the other years as well.

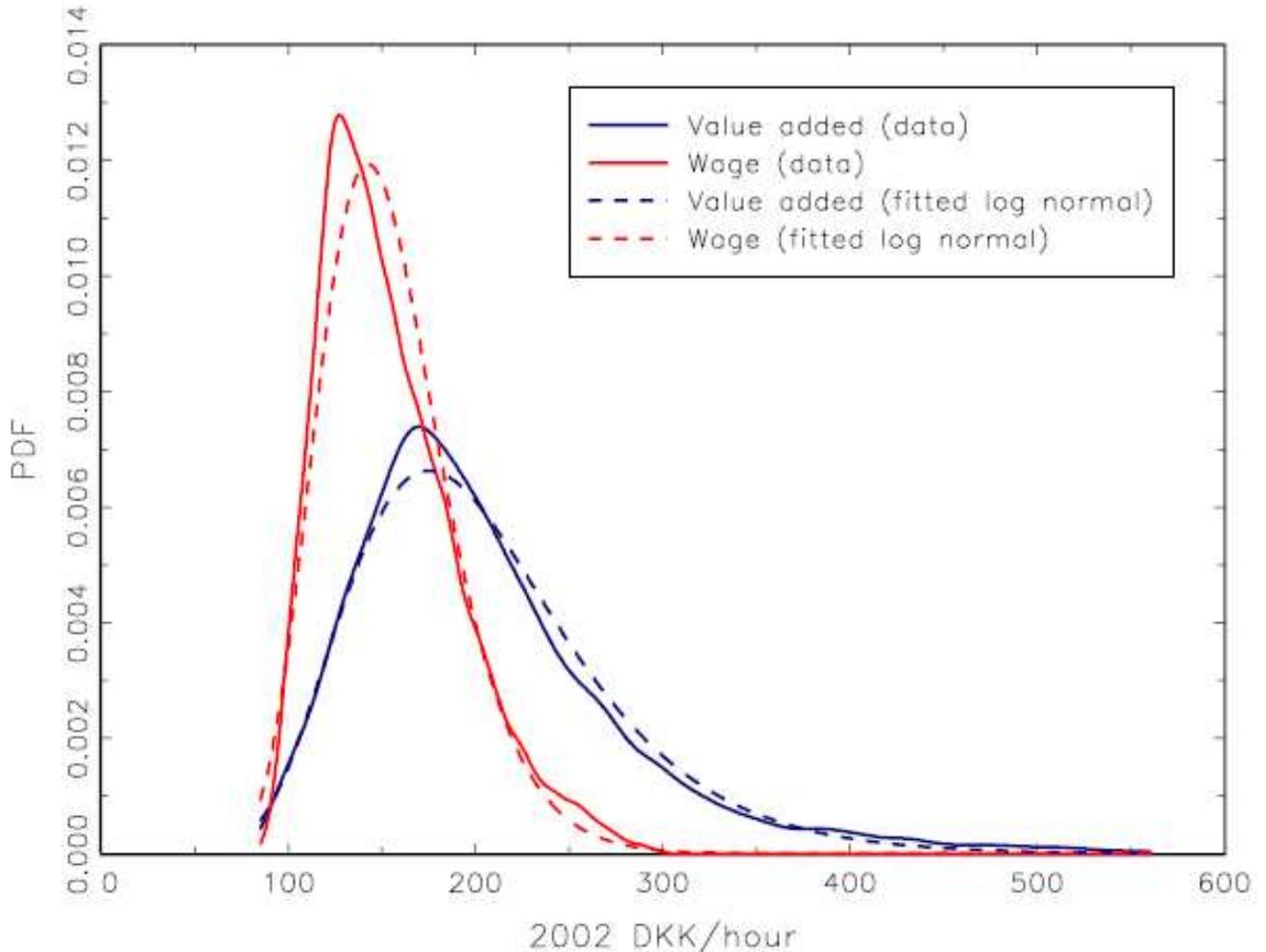


Figure 1: Danish Firm Wage and Productivity Distributions

productivity exceeds the average wage paid, as one should expect.

In Figure 2, the non-parametric regression and OLS regression of firm labor productivity on average firm wage cost is plotted. This evidence again suggests that the relationship is close to linear for the bulk of the mass of firms in the data but may be weakly concave at high levels of productivity.

The strong positive relationship between firm wage and productivity supports "rent-sharing" theories of the wage determination. Of course, the linear nature of the relationship for the bulk of the observations is implied by the simple bilateral bargaining model assumed in this paper. But, there are alternative models of rent sharing that might be incorporated instead. Although the dynamic monopsony model proposed by Burdett and Mortensen (1998) implies a concave relationship, Mortensen (2003) shows that the degree of concavity observed is not nearly as large as that implied by dynamic monopsony. Postel-Vinay and Robin (2002) and Cahuc, P., F. Postel-Vinay, and J-M Robin (2006) develop and estimate a structural model in which the growth in worker wages with experience can be attributed

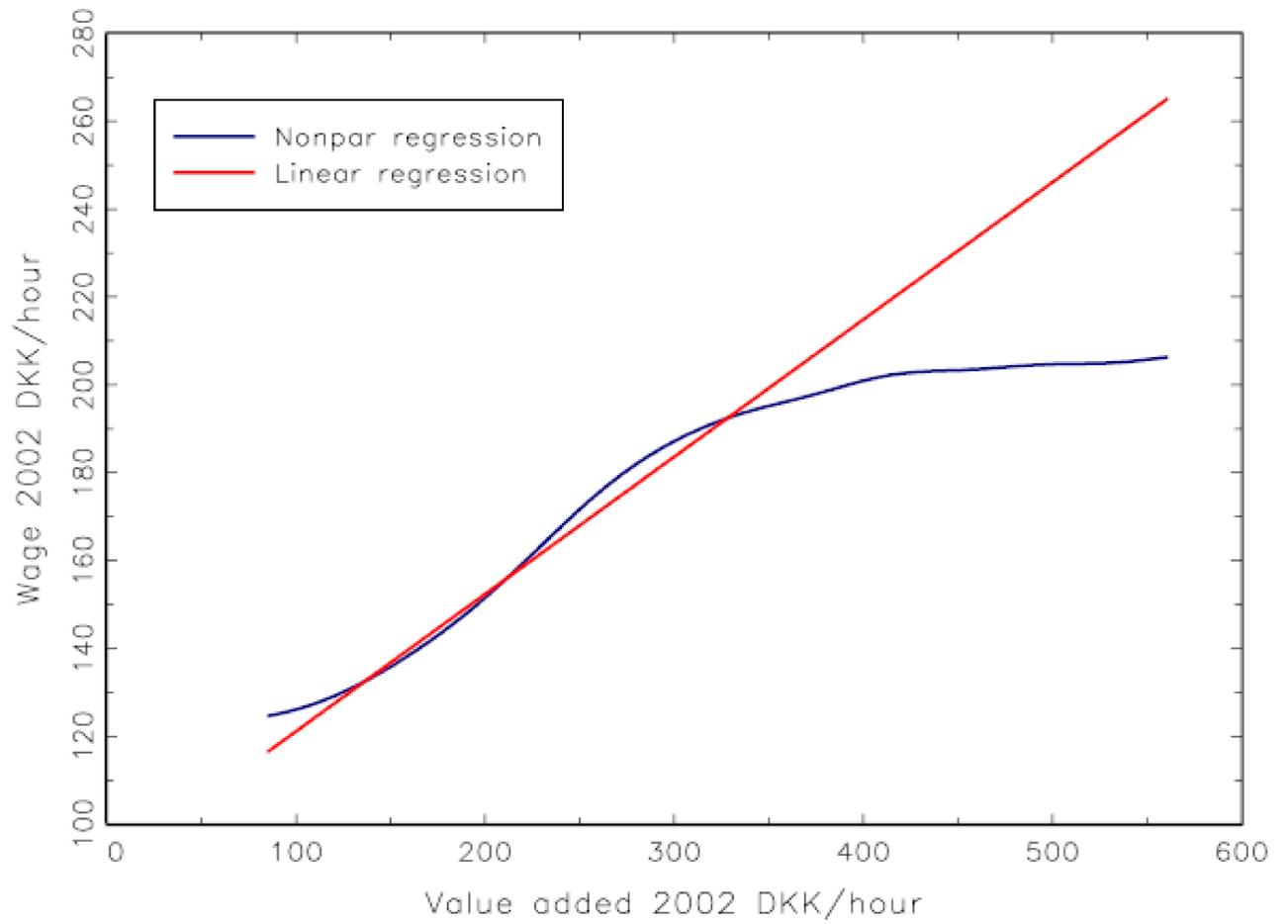


Figure 2: Danish Firm Wage vs Firm Productivity

to Bertrand competition between a worker's current and a potential alternative employer. Although this wage setting process is more complicated than either bilateral bargaining or dynamic monopsony, it may also be consistent with the quantitative relationships reported here.

### 3 The Model

#### 4 Household Preferences

There is a continuum  $L$  of identical households each composed of a continuum of individuals. Each household chooses final consumption and leisure time paths to maximize intertemporal utility. There is a single final good which can be used for either consumption or investment which is produced in the market sector with a continuum of intermediate goods. Labor, both in the market and at home, is used to produce intermediate goods. Home production requires one unit of labor input per unit of intermediate good output. Leisure is equal to the available labor force employed in neither the market nor home production. The instantaneous utility function is of the linear form  $C - b(e + h)$  where  $e$  and  $h$  represent the fractions of the individuals in the household who are employed in the market and at home respectively and  $b$  is the cost of employment in terms of consumption of final output. Households discount future utility at the fixed rate  $r$  which is also equal to the equilibrium rate of interest. The labor supply constraint is  $1 - h - e \geq 0$ .

##### 4.1 Derived Demand, Prices, and Wages

The single final good is supplied by many competitive suppliers and the aggregate quantity produced is determined by a continuum of intermediate inputs. Final output at any point in time  $t$  is determined by the following Cobb-Douglas production function with constant returns to scale:

$$\ln Y = \ln(AK) + \int_0^K \frac{1}{K} \ln(x(j)) dj \quad (1)$$

where  $x(j)$  is the quantity of intermediate product  $j$  available. As the final good is supplied by a competitive sector, the profit maximizing demand for each input given the final output production rate is

$$x^d(j) = \frac{PY}{p(j)K} \quad (2)$$

where  $p(j)$  is the price of input  $j$  and  $P$ , the price of the final good, is determined by

$$\ln(AP) = \int_0^K \frac{1}{K} \ln p(j) dj. \quad (3)$$

Labor is the only input in the production of any intermediate good variety. Although a intermediate good can be produced by a competitive fringe, its creator has a cost advantage. The labor requirement of the fringe, interpreted as the households, is unity as previously stated but the productivity of a worker employed by the creator of the good, denoted by  $q$ ,

generally exceeds unity. The marginal rate of substitution between final output and labor,  $b_t$ , represent the cost of home production of any intermediate good per unit of the final good. An innovator captures the market for any good or service created by setting its price just below the production cost of the competitive fringe. That is the price of any intermediate good is  $p(j) = Pb$  for all  $j$ . Hence, equation(3) implies

$$b = \frac{p(j)}{P} = A, \text{ for all.} \quad (4)$$

We assume that the real wage is the outcome of bargaining over current output where an employed worker must quit in order to engage in negotiation with an alternative employer and the worker's default option while negotiating is either home production. In other words, neither side can commit to sharing future income and the worker cannot search while engaged in negotiation. Suppose that rounds of bargaining take place such that the worker makes the offer and the employer accepts or reject in each with probability  $\beta \in (0, 1)$  and where the roles are reversed with complementary probability  $1 - \beta$ .

Provided that match surplus is positive, bargaining continues until an acceptable offer is made. Under complete information, it is well known that the unique sub-game perfect equilibrium exists in which agreement takes place after the first round. Since each side make an offer that is equivalent to continuing the game and other accepts, then the expected real wage for any intermediate good supplier of productivity  $q$  is

$$w(q) = b + \beta (p(j)q/P - b) = (1 - \beta + \beta q)A \quad (5)$$

when the length of a bargaining round is negligible where the second equality follows from (4).

Because the prices of the final and intermediate goods are all equal, the derived demand for workers for product  $j$  produced by a line of productivity  $q$  is

$$n_j^d(q) = \frac{x^d(j)}{q} = \frac{Y}{AKq} \quad (6)$$

from equations (2), where  $q$  represents the labor productivity of the supplier of the intermediate good,  $Y$  is the demand for the final good, and  $K$  is the measure of intermediate product lines in the market.

Although there is no current marginal product to divide if the labor force available is greater than or equal to  $n^d(q)$ , the employer still may have demand for another worker or two to protect against quit risk. In principal, the employer need only pay the reservation wage to hire the marginal worker for this purpose. However, this form of wage discrimination may be either impossible or very expensive in terms of internal conflict. For these reasons, we assume that the employer pays the same wage,  $w(q)$ , per additional worker hired up to some maximal number of employee, denoted  $\bar{n}(q)$ , which is chosen optimally given this constraint. Hence, equations (5) and (4) imply that net current revenue is given by

$$\pi(q, n, n^d) = \begin{cases} [p(j)q/P - w(q)] n = (1 - \beta)(q - 1)An & \text{if } n \leq n^d \\ = [p(j)q/P - w(q)] n^d - w(q)(n - n^d) \\ = qAn^d - (1 - \beta + \beta q)An \end{cases} \quad (7)$$

## 4.2 Worker Search

Our model of labor search friction and the turnover is taken from Christensen et. al (2005). Workers are homogenous with respect to both market and home production. Worker's search with an endogenous intensity or effort  $s$  to maximize the present value of expected future labor income. Formally, the rate at which workers contact job openings as a consequence of search effort is  $\delta_1 + \lambda(\theta)s$ , where  $\lambda(\theta)$  is a search efficiency parameter that depends on "market tightness" parameterized by  $\theta$  and  $\delta_1$  is an "exogenous" component of the arrival rate capturing other reasons for transitions from one job to another. The cost of search expressed in units of output, denoted  $c_s(s)$ , is increasing and strictly convex. Finally, let  $\delta_0$  represent the rate at which workers are laid off.

Of course, a worker's choice of search intensity depends on incentives and these differ with employment status. When unemployed, a worker has the option of home production which generates an implicit income equal to  $b$ . Hence, current income expressed in term of final output when unemployed net of search cost is  $b - c_s(s)$  and the value of unemployment expressed in units of final output,  $U$ , satisfies the continuous time Bellman equation

$$rU = \max_{s \geq 0} \left\{ b - c_s(s) + (\delta_1 + \lambda s) \int (\max \langle W(w), U \rangle - U) dF(w) \right\}$$

where  $\lambda$  is as search efficiency parameter and  $F(w)$  is the wage offer c.d.f., the fraction of employer contact made that offer wage  $w$  or less. The last term is the expected change in value associated with finding a job.

As an employed workers moves from lower to higher paying employment when the opportunity arises, the value of employment at a wage  $w$ , also expressed in terms of final output, solves the following Bellman equation:

$$rW(w) = \max_{s \geq 0} \left\{ \begin{array}{l} w - c_s(s) + \lambda s \int (\max \langle W(x), W(w) \rangle - W(w)) dF(x) \\ + \delta_1 \int (\max \langle W(x), U \rangle - W(w)) dF(w) + \delta_0(U - W(w)) \end{array} \right\}.$$

where  $\delta_0$  is the job destruction rate and  $w$  is the real wage expressed in units of final output. Note that worker's do not choose between the current and new job in the case of "exogenous reallocation." The idea is that the transition has taken place for other reason's not captured by the wage differential between the original and new employment opportunity.<sup>4</sup>

Under the assumption that the cost of search and the search efficiency parameter is the same whether employed or not, the real reservation wage,  $R$ , is simply equal to the flow value of home production. Formally,

$$W(R) = U \implies R = b = A \tag{8}$$

where the last equality is implied by equation (4). Obviously, in equilibrium, all offered wages will be acceptable by equation (5) and the fact that  $q > 1$ .

Note that

$$W'(w) = \frac{1}{r + \delta_0 + \delta_1 + \lambda s(w)[1 - F(w)]} > 0$$

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<sup>4</sup>Both Jolivet et al. (2004) and Christensen et al. (2005) find that "exogenous" reallocation is needed to fit the data on job separation flows.

which implies

$$\begin{aligned}
& \int (\max \langle W(x), W(w) \rangle - W(w)) dF(w) \\
&= \int_w^{\bar{w}} [W(x) - W(w)] dF(w) \\
&= \int_w^{\bar{w}} \left( \frac{1 - F(x)}{r + \delta_0 + \delta_1 + \lambda s(x)[1 - F(x)]} \right) dx
\end{aligned}$$

where the second equality follows by integration by parts. Consequently, the optimal search effort strategy is defined by

$$s(w) = \arg \max_{s \geq 0} \left\{ \left( \lambda \int_w^{\bar{w}} \left( \frac{1 - F(x)}{r + \delta_0 + \delta_1 + \lambda s(x)[1 - F(x)]} \right) dx \right) s - c_s(s) \right\} \quad (9)$$

where the intensity when unemployed is the solution when  $w = R = b$ , denoted  $s(b)$ . That is, optimal search effort maximizes the difference between the total return and cost of search. As a consequence, the first order condition for an interior solution implies that the solution for the optimal search strategy satisfies the integral equation

$$c'_s(s(w)) = \lambda \int_w^{\bar{w}} \left( \frac{1 - F(x)}{r + \delta_0 + \delta_1 + \lambda s(x)[1 - F(x)]} \right) dx. \quad (10)$$

The solution is unique for every choice of  $\theta$  and  $F(w)$  with the property that  $s(\bar{w}) = 0$  given that a differentiation with respect to  $w$  yields an ordinary first order differential equation in  $s(w)$ .

The search efficiency parameter  $\lambda$  depend on the inputs into the labor market matching process, aggregate search and recruiting effort. Given a matching function that is increasing, concave, and homogenous of degree one in the total two aggregates, the job finding rate per unit of search effort,  $\lambda(\theta)$ , is increasing and concave where "market tightness,"  $\theta$ , is the ratio of the aggregate recruiting to aggregate search effort. This variable, which is endogenous to the market, will be defined more precisely after the employer's recruiting effort decision problem is fully specified.<sup>5</sup>

### 4.3 Firm Dynamics

The productivity of a product line,  $q_j$ , is a random variable realized when the product is created. Let  $\Gamma_\tau(q), \tau \in \{1, 2, \dots, T\}$ , denote the cumulative distribution of productivity and size for any firm of type  $\tau$ . We assure that the magnitude of the type index reflects the productivity rank of the type in the sense of first order stochastic dominance. That is,  $\tau > \tau'$  implies that product lines of firms of  $\tau$  are likely to be more productive than firms of type  $\tau'$ , i.e.,  $\Gamma_\tau(q) \leq \Gamma_{\tau'}(q)$  for all  $q \in [1, \bar{q}]$ .

Firm size as reflected in the number of products supplied is generated by a research and development process as introduced in Klette and Kortum (2004). At any point in time, a

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<sup>5</sup>For details on the concept of a matching function and its applications, see Pissarides (2000) and Petrongolo and Pissarides (2001).

firm has  $k$  product lines and can grow only by creating new product varieties. Investment in R&D is require to create new products. Specifically, the firm's R&D investment flow generates new product arrivals at frequency  $\gamma k$  where  $\gamma$  represents the firm's innovation rate per product line. The total R&D investment cost expressed in terms of output is  $c(\gamma)k$  where  $c(\gamma)$  is assumed to be strictly increasing and convex function. The assumption that the total cost of R&D investment is linearly homogenous in the new product arrival flow,  $\gamma k$ , and the number of existing product,  $k$ , "captures the idea that a firm's knowledge capital facilitates innovation," in the words of Klette and Kortum (2004). The specification assumption also implies that Gibrat's law holds in the sense that innovation rates are size independent contingent on type, a property needed to match the data on firm growth. Finally, every product is subject to destruction risk with exogenous frequency  $\delta_0$ . Given this specification, the number of products supplied by any firm is a stochastic birth-death process characterized by "birth rate"  $\gamma$  and "death rate"  $\delta_0$ .

Workers are acquired through a matching process and workers quit to take better paying jobs. The rate at which workers employed in a product line of productivity  $q$  quit is equal to

$$\varphi(q) = \delta + \lambda(\theta)s(w(q))[1 - F(w(q))] \quad (11)$$

where  $\delta_1$  represents the rate of "exogenous reallocation." By assumption, an employer meets a worker type with probability proportional to the frequency with that type contacts offers under random matching. As an unemployed workers accepts any wage, and an employed searching worker accepts only if the wage offered exceeds that currently paid, the probability that the wage  $w$ , is accepted by a worker met at random is

$$\rho(q) = \frac{(1 - e) (\delta_1 + s(R)\lambda(\theta)) + e \left( \delta_1 + \lambda(\theta) \int_R^{w(q)} s(x)dG(x) \right)}{(1 - e) (\delta_1 + s(R)\lambda(\theta)) + e \left( \delta_1 + \lambda(\theta) \int_R^{\bar{w}} s(x)dG(x) \right)}. \quad (12)$$

where  $G(w)$  is the fraction of worker employed at wage  $w$  or less.

Immediate and costless intra-firm reallocation of workers is not possible by assumption. Hence, each product line hires and fires independently. At any point in time, the employment state of any product line is represented by the current number of employees, denoted as an integer  $n$ . The rate at which workers contacted is proportional to the firm's recruiting effort, denoted by  $v$ . Let  $c_v(v)$ , an increasing convex function, represent the cost of recruiting effort. Since  $\rho(q)$  accept a job offered by a product line of productivity  $q$ , the frequency with which workers are hired is  $\eta\rho(q)v$  where  $\eta$  is the worker contact rate per unit of recruiting effort. Finally, the frequency with which quits occur is  $\varphi(q)n$ .

#### 4.4 Firm Decisions and Value

The state of a firm at any point in time is described by the tuple  $(\vec{n}, \vec{q}, \vec{\alpha}, k)$  where  $\vec{n} = (n_1, n_2, \dots, n_k)$  is a vector that specifies the numbers of workers available for the production of each of the  $k$  intermediate goods supplied, and  $\vec{q} = (q_1, q_2, \dots, q_k)$  is a vector representing the realized productivities with which the products are produced. The firm's product creation rate,  $\gamma$ , and vector of search efforts  $\vec{v} = (v_1, v_2, \dots, v_k)$  are chosen to maximize the expected present value of future profit contingent on the current state.

The value of a firm of type  $\tau$  in any state,  $\tilde{V}_\tau(\vec{n}, \vec{q}, k)$ , is the solution to the continuous time Bellman equation

$$\begin{aligned}
& r\tilde{V}_\tau(\vec{n}, \vec{q}, k) \\
= & \sum_{j=1}^k \pi(q_j, n_j, n_j^d(q_j)) \\
& + k \max_{\gamma \geq 0} \left\{ \gamma \int \left[ \tilde{V}_\tau((\vec{n}, 0), (\vec{q}, q), k+1) - \tilde{V}_\tau(\vec{n}, \vec{q}, k) \right] d\Gamma_\tau(q) - c_\gamma(\gamma) \right\} \\
& + \delta_0 k \left( \frac{1}{k} \sum_{j=1}^k \left[ \tilde{V}_\tau(\vec{n}_{\langle j \rangle}, \vec{q}_{\langle j \rangle}, k) - \tilde{V}_\tau(\vec{n}, \vec{q}, k) \right] \right) \\
& + \sum_{j=1}^k \max_{v \geq 0} \left\{ \eta \rho(q_j) v \left( \tilde{V}_\tau((n_1, \dots, n_j + 1, \dots, n_k), \vec{q}, k) - \tilde{V}_\tau(\vec{n}, \vec{q}, k) \right) - c_v(v) \right\} \\
& + \sum_{j=1}^k \varphi(q_j) n_j \left[ \tilde{V}_\tau(n_1, \dots, n_j - 1, \dots, n_k), \vec{q}, k) - \tilde{V}_\tau(\vec{n}, \vec{q}, k) \right]
\end{aligned} \tag{13}$$

where  $\vec{n}_{\langle j \rangle}$  is vector of size  $k - 1$  created from  $\vec{n}$  by dropping its  $j$ th element and. The first term on the right is simply the net revenue flow from its current product portfolio. The second term, given the optimal choice of the innovation rate, is the maximal value of its R&D operation, the difference between the product of its innovation rate and the expected capital gain realized when a new product line is created and the expected flow cost of innovation. The third term is the expected capital loss in value attributable to the destruction of an existing product. The fourth term accounts for the change in value associated the possibility than an additional worker is added to the labor force of any product line, given that recruiting effort is chosen optimally, and the last term accounts for the expected loss in value attributable to the possibility that any employee quits.

**Proposition 1** *The value function for the firm's problem takes the form*

$$\tilde{V}_\tau(\vec{n}, \vec{q}, k) = \sum_{j=1}^k V_{n_j}(q_j, n_j^d(q_j)) + k\Psi_\tau(q), \tag{14}$$

where the function  $n_j^d(q) = \alpha_j Y / Aq$ , as defined in equation (6) and  $\Psi_\tau(q)$  solves

$$(r + \delta_0) \Psi_\tau = \max_{\gamma \geq 0} [\gamma (E_\tau V_0(\tilde{q}, Y / KA\tilde{q}) + \Psi_\tau) - c_\gamma(\gamma)], \tag{15}$$

where  $E_\tau(\cdot)$  represents the expectation taken with respect to the joint distribution of  $q$  and  $\alpha$ , and  $V_n(q, n^d)$  is a solution to the difference equation

$$\begin{aligned}
(r + \delta_0) V_n(q, n^d) & = \pi(q, n, n^d) + \max_{v \geq 0} \left\{ \eta(\theta) \rho(q) v (V_{n+1}(q, n^d) - V_n(q, n^d)) - c_v(v) \right\} \\
& + \varphi(q) n [V_{n-1}(q, n^d) - V_n(q, n^d)], \quad n \geq 0.
\end{aligned} \tag{16}$$

**Proof.** Insert the conjecture into equation (13) to obtain

$$\begin{aligned}
& r \sum_{i=1}^k [V_n(q_j, n_j^d(q_j)) + \Psi_\tau] \\
= & \sum_{j=1}^k \pi(q_j) \min \langle n_j, n^d(q_j) \rangle + k \max_{\gamma \geq 0} [\gamma (E_\tau V_0(\tilde{q}, Y/KA\tilde{q}) + \Psi_\tau) - c(\gamma)] - \delta_0 \sum_{j=1}^k [V_{n_j}(q_j, n_j^d(q_j)) + \Psi_\tau] \\
& + \sum_{j=1}^k \max_{v \geq 0} 0 \{ \eta \rho(q_j) v (V_{n_j+1}(q_j, n_j^d(q_j)) - V_n(q_j, n_j^d(q_j))) - c_v(v) \} \\
& + \sum_{j=1}^k \varphi(q_j) n_j [V_{n-1}(q_j, n_j^d(q_j)) - V_n(q_j, n_j^d(q_j))] \\
= & \sum_{j=1}^k \left\{ \begin{array}{l} \pi(q_j, n_j, n_j^d(q_j)) \\ + \max_{v \geq 0} \{ \eta \rho(q_j) v (V_{n_j+1}(q_j, n_j^d(q_j)) - V_n(q_j, n_j^d(q_j))) - c_v(v) \} \\ + \varphi(q_j) n_j [V_{n_j-1}(q_j, n_j^d(q_j)) - V_{n_j}(q_j, n_j^d(q_j))] \end{array} \right\} \\
& + k \left( \max_{\gamma \geq 0} [\gamma (E_\tau V_0(\tilde{q}, Y/KA\tilde{q}) + \Psi_\tau) - c(\gamma)] - \delta_0 \Psi_\tau \right).
\end{aligned}$$

Hence, equations (14), (15) and (16) are verified. As the solution to (13) can be represented as the fixed point of a contraction map, there is only one solution of this form. ■

The value function  $V_n(q, n^d)$  is the expected present value of the future profit accruing to an existing product line of productivity  $q$  with derived demand  $n^d$  and  $n$  employees, while  $\Psi_\tau$  is the value of the option to create another product line associated with an existing product.

## 4.5 Recruiting Effort

Obviously, equation (2) implies that the optimal recruiting effort per product line conditional on the number of current employees and their productivity is

$$v_n(q, n^d) = \arg \max_{v \geq 0} \{ \eta(\theta) \rho(q) v [V_{n+1}(q, n^d) - V_n(q, n^d)] - c_v(v) \}. \quad (17)$$

By implication, the maximum labor force size,  $\bar{n}(q)$ , is the first integer for which the marginal value of another worker is strictly negative. In other words, it satisfies

$$\begin{aligned}
V_{\bar{n}}(q, n^d(q)) - V_{\bar{n}-1}(q, n^d(q)) & \geq 0 \\
V_{\bar{n}+1}(q, n^d(q)) - V_{\bar{n}}(q, n^d(q)) & < 0.
\end{aligned} \quad (18)$$

**Proposition 2** *A unique strictly non-negative, continuous, and increasing solution for the value of a product line with  $n$  workers,  $V_n(q, n^d)$ , exists if  $q > 1$ ,  $c_v(v)$  is strictly increasing, convex and  $c(0) = c_v(0) = 0$ . Furthermore,  $V_{n+1}(q, n^d) - V_n(q, n^d)$  is non-negative and is strictly increasing in  $q$  for all  $n \leq n^d(q)$ .*

**Proof.** First, rewrite equation (16) in the following equivalent form.

$$V_n(q, n^d) = \max_{v \geq 0} \left\{ \frac{\pi(q, n, n^d) + \varphi(q) n V_{n-1}(q, n^d) + \eta \rho(q) v V_{n+1}(q, n^d) - c_v(v)}{r + \delta_0 + \varphi(q) n + \eta \rho(q) v} \right\} \quad (19)$$

The right hand side of (19) defines a map from the set of positive bounded functions that are continuous in  $q$  and  $n^d$  and increasing in  $n^d$  into itself. Because one can verify Blackwell's sufficient conditions for a contraction, a unique positive solution with this properties exists. Specifically, the map is obviously increasing and is of modulus

$$0 < \frac{\varphi(q)n + \eta\rho(q)v}{r + \delta_0 + \varphi(q)n + \eta\rho(q)v} \leq \frac{\varphi(q)n^d + \eta\rho(q)\bar{v}(q, n^d)}{r + \delta_0 + \varphi(q)n^d + \eta\rho(q)\bar{v}(q)} < 1$$

where

$$0 \leq \bar{v}(q, n^d) = \arg \max_{v \geq 0} \left\{ \eta\rho(q)v \left( \frac{\pi(q, n^d, n^d)}{r + \delta_0} \right) - c_v(v) \right\} < \infty$$

is an upper bound on the optimal choice of recruiting effort for any  $q$  given that  $c_v(v)$  is strictly convex by assumption,  $\pi(q, n, n^d) \leq \pi(q, n^d, n^d)$  and equation (7), and  $V_{n+1}(q, n^d) - V_n(q, n^d) \leq \frac{\pi(q, n^d, n^d)}{r + \delta_0}$  for all  $n$  given that values are positive and bounded by  $\frac{\pi(q, n^d, n^d)}{r + \delta_0}$ . In addition, the fixed point has the property that  $V_n(q, 0) = 0$  when  $n^d(q) = 0$  and  $V_n(q, n^d) > 0$  if  $n^d(q) \geq 1$  and  $q > 1$ .

That the solution  $V_n(q, n^d)$  is also increasing in  $q$  given  $n^d$  is implied by equation (2),  $\rho'(q) > 0$  and  $\varphi'(q)$  if  $V_{n+1}(q, n^d) - V_n(q, n^d)$  is increasing in  $q$  and nonnegative for all  $n^d$ . By differencing both sides of equation (16), one obtains the following expression relating changes in value associated with a change in the number of workers employed

$$\begin{aligned} & V_{n+1}(q, n^d) - V_n(q, n^d) & (20) \\ = & \frac{\left[ \begin{aligned} & [\min \langle n+1, n^d(q) \rangle - \min \langle n, n^d(q) \rangle] \pi(q, n, n^d) + c_v(v_n(q)) \\ & + \max_{v \geq 0} \{ \eta(\theta)\rho(q)v (V_{n+2}(q, n^d) - V_{n+1}(q, n^d)) - c_v(v) \} \\ & + \varphi(q)n [V_n(q, n^d) - V_{n-1}(q, n^d)] \end{aligned} \right]}{r + \delta_0 + \varphi(q)(n+1) + \eta(\theta)\rho(q)v_n(q)} \end{aligned}$$

Now, suppose that the maximal labor force size exceed derived demand, i.e.,  $\bar{n}(q) \geq n^d(q)$  for any  $q$ . Given  $n^d(q) \geq 1$ , equations (7) and (17), we show that equation (20) implies that  $V_{n+1}(q, n^d) - V_n(q, n^d) > 0$  for all  $n-1 \leq n^d(q)$  by induction. First, note that  $V_1(q) - V_0(q) > 0$  if  $n^d(q) \geq 1$ . Since the right side of (17) is always non-negative and profit is non-negative from equation (7), the supposition that  $V_n(q) - V_{n-1}(q) > 0$  implies  $V_n(q) - V_{n-1}(q) > 0$  for all  $n-1 \leq n^d(q)$ , a fact that competes the induction proof. Of course, this result also confirms the supposition  $\bar{n}(q) \geq n^d(q)$ .

Finally, the fact that the right hand side of (20) maps the set of value function differences that are continuous and increasing  $q$  into itself given equation (7) and  $V_n(q, n^d) - V_{n-1}(q, n^d) > 0$  for all  $n < n^d(q)$  and the fact that the first term on the right side of (20) is zero when  $n \geq n^d(q)$  implies that the fixed point of the contraction also has this property if the partial derivative of the right hand side is increasing in the probability of acceptance,  $\rho(q)$  and decreasing in the separation rate  $\varphi(q)$ . As this condition is satisfied so long as the difference is non-negative,  $V_n(q, n^d)$  is increasing in  $q$  for all  $n^d$  by equation (16). ■

**Corollary 3** *Optimal recruiting effort  $v_n(q, n^d)$  is continuous in both arguments and is increasing in  $q$  for all  $n \leq n^d$ .*

## 4.6 Product Innovation

Equation (15) defines the value of innovation activity per product line for a type  $\tau$  firm. It is the present value of the maximal difference between two part, the product of the return to innovation and the chosen innovation rate less the R&D investment required to sustain that rate. The return is the sum of two parts, the expected present value of the profit accruing to the creation of a new product plus that value of the opportunity to create another product facilitated by the existence of a new product line, the value of the knowledge embodied in the creation of a new product. From equation (15), the optimal innovation strategy is

$$\gamma_\tau = \arg \max_{\gamma \geq 0} \{ (E_\tau V_0(q, n^d(q)) + \Psi_\tau) \gamma - c_\gamma(\gamma) \}. \quad (21)$$

Note that the choice is independent of both the number of products currently supplied and of the employment vector.

**Proposition 4** *If the cost of R&D,  $c_\gamma(\gamma)$ , is increasing, strictly concave, and  $c_\gamma(0) = c'_\gamma(0) = 0$ , and final output per input good ratio  $Y/K$  is sufficiently small, then the optimal product creation rate is positive, less than the job destruction rate, and increasing in  $Y/K$ .*

**Proof.** If a solution to (21) exists, then the value of an additional product line is defined by

$$\Psi_\tau = \max_{\gamma \geq 0} \left\{ \frac{\gamma E_\tau V_0(\tilde{q}, Y/AK\tilde{q}) - c_\gamma(\gamma)}{r + \delta_0 - \gamma} \right\}. \quad (22)$$

Hence, the first order condition for the optimal innovation rate can be written as

$$f(\gamma) = (r + \delta_0 - \gamma) (E_\tau V_0(\tilde{q}, Y/AK\tilde{q}) - c'_\gamma(\gamma)) + \gamma E_\tau V_0(\tilde{q}, Y/AK\tilde{q}) - c_\gamma(\gamma) = 0 \quad (23)$$

and the second order condition requires  $f'(\gamma) = -(r + \delta_0 - \gamma) c''(\gamma) \leq 0$  at a maximal solution. As  $f'(0) = (r + \delta_0) E_\tau V_0(\tilde{q}, Y/AK\tilde{q}) > 0$ , the first order condition has a unique solution satisfying  $0 < \gamma < \delta_0$  and the sufficient second order condition is satisfied if  $f'(\delta_0) = (r + \delta_0) E_\tau V_0(\tilde{q}, Y/AK\tilde{q}) - r c_\gamma(\delta_0) - c_\gamma(\delta_0) < 0$ . As we have also shown that  $V_0(q, n^d)$  is a continuous increasing function of  $n^d$  with the property that  $V_0(q, 0) = 0$ , the fact that  $n^d(q) = Y/AKq \leq Y/AK$  establishes the claim that there is a unique solution in the desired interval for all  $Y/K$  sufficiently small. ■

## 4.7 Entry

Assume that entry requires a successful innovation, that the cost of innovation by a potential entrant is  $c(\gamma)$ , and that firm type is unknown to an entrepreneur prior to entry. The entry rate is the product  $v = m\gamma_0$  where  $m$  is a given measure of entrepreneurs and  $\gamma_0$  is the frequency with which any one of them creates new product. As the optimal innovation rate by a potential entrant maximizes expected value, equal to  $\gamma_0 \sum_\tau [E_\tau V(0, q) + \Psi_\tau] \phi_\tau - c(\gamma_0)$  where  $\phi_\tau$  is the exogenous probability of being a type  $\tau$  firm, the optimal choice is defined

$$v = m\gamma_0 = m \arg \max_{\gamma \geq 0} \left\{ \left( \sum_\tau [E_\tau V_0(\tilde{q}, Y/AK\tilde{q}) + \Psi_\tau] \phi_\tau \right) \gamma - c_\gamma(\gamma) \right\}. \quad (24)$$

**Proposition 5** *The innovation rate  $v$  is increasing in  $Y/K$  for all  $q$ .*

**Proof.** The claim is implied by equation (24) and the fact that  $V_0(q, n^d)$  is increasing in  $n^d$ .  
 ■

## 5 Market Equilibrium

### 5.1 The Steady State Distributions of Products, Offers, and Wages

Unemployed workers find jobs at the rate  $\delta_1 + s(b)\lambda(\theta)$  and lose them at the product destruction rate  $\delta_0$ . The fraction unemployed in any household is  $1 - e$ . Hence, the steady state employment rate,  $e$ , that which balances inflow and outflow, is the solution to

$$\frac{1 - e}{e} = \frac{\delta_0}{\delta_1 + \lambda(\theta)s(b)}. \quad (25)$$

As the wage paid is strictly monotone in product line productivity, the steady state condition for the distribution of market employment over productivity requires that the flow into the stock of those employed at wage  $w(q)$  or less equals the flow out. As unemployed workers and workers reallocated for exogenous reason flows into the stock earning  $w$  or less with probability  $F(w)$  while those who quit to take a job paying more than  $w$  and those that either lose their jobs or take another job for exogenous reasons flow out,

$$[\delta_1 + \lambda(\theta)s(b)](1 - e)F(w(q)) + \delta_1 G(q)eF(w(q)) = e \left[ (\delta_0 + \delta_1)G(w(q)) + \lambda(\theta)[1 - F(w(q))] \int_1^q s(x)dG(x) \right]$$

where  $G(q)$  is the fraction of workers employed in a product line of productivity  $q$  or less. Hence, this fact and the steady state unemployment condition, equation (25), yield the following relationship between the wage and offer distribution functions:

$$F(w(q)) = \frac{(\delta_0 + \delta_1)G(q) + \lambda(\theta) \int_1^q s(w(x))dG(x)}{\delta_0 + \delta_1 G(q) + \lambda(\theta) \int_1^q s(w(x))dG(x)}. \quad (26)$$

Obviously, this equation defines a mapping from the space of wage distributions to the space of offer distributions. Since a differentiation of both side yields an ordinary first order differential equation in  $G(q)$  for any given  $F(w(q))$  and  $F(b) = G(b)$ , the converse is also true. In sum, the mapping is one-to-one and continuous in  $\theta$  given that  $\lambda(\theta)$  is continuous.

The fact that the demand for intermediate goods must equal the supply determines the rate at which final output is produced. The total demand is  $\int PY/K p_j dj = Y/A$  from equation (4). Because productivity at home is unity, all those not employed in the market are willing to engage in home production in equilibrium ( $h = 1 - e$ ), and market producers supply the quantity  $qn$  in each product line of productivity  $q$  with  $n$  employees provided that employment is no greater than demand,  $n \leq n^d(q)$ , the equality of supply and demand requires

$$Y = AL \left( 1 - e + e \sum_{\tau} K_{\tau} \left( \int_1^{\bar{q}} \sum_{n=0}^{\hat{n}^d(q)} qn P_n(q) + q [n^d(q) - \hat{n}^d(q)] P_{\hat{n}^d+1}(q) \right) d\Gamma_{\tau}(q) \right)$$

where  $\hat{n}^d(q)$  is the largest integer such that  $\hat{n}^d(q) \leq n^d(q)$ ,  $L$  is the measure of households in the economy,  $K_\tau$  is the measure of products supplied by firms of type  $\tau$  and  $P_n(q)$  is the fraction of product lines of productivity  $q$  that employ  $n$  workers. We have already shown that employers generally hoard workers in the sense that the maximum labor force size is at least as large as derived demand ( $\bar{n}(q) \geq n^d(q)$ ). Hence, if  $n^d$  is not an integer then, the residual demand  $n^d - \hat{n}^d$  is supplied by one of these workers when available. As a consequence,  $Y$  is continuous in  $\hat{n}^d$  and, therefore,  $n^d$ .

Note that equations (??) and (??) imply the following upper bound on final good output

$$\bar{Y} = AL\bar{q} \quad (27)$$

given the labor supply constraint ( $e + h \leq 1$ ).

Under the assumption that searching workers meet employers with probabilities proportional to recruiting error, the probability that any searching worker is offered a wage less than or equal to  $w(q)$  is the fraction of aggregate recruiting investment of those employers offering these wages. Since  $w(q)$  is strictly increasing in  $q$ ,

$$F(w(q)) = \frac{\sum_\tau^T K_\tau \int_1^q \sum_{n=0}^\infty v_n(x, Y/AKx) P_n(x, Y/K) d\Gamma_\tau(x)}{\sum_\tau^T K_\tau \int_1^q \sum_{n=0}^\infty v_n(x, Y/AKx) P_n(x, Y/K) d\Gamma_\tau(x)} \quad (28)$$

where  $v_n(q, n^d(q))$  is the recruiting effort of a product line of productivity  $q$  with  $n$  employees.

The measure of products supplied by the set of type  $\tau$  firms evolves according to the law of motion,  $\dot{K}_\tau = v\phi_\tau + \gamma_\tau K_\tau - \delta_0 K_\tau$  where  $v$  is the innovation rate of new entrants,  $\phi_\tau$  is the fraction of entrant who are of type  $\tau$ , and  $\gamma_\tau$  is the innovation rate of type  $\tau$  firms per product line. In other words, the net rate of change in the measure is equal to the sum of the flows of products supplied by new entrants and continuing firms respectively less the flow of product lines currently supplied by the type that are destroyed. Hence, in steady state, the measure of products supplied by type  $\tau$  firms and the aggregate measure of product are

$$K_\tau = \frac{v\phi_\tau}{\delta_0 - \gamma_\tau} \text{ and } K = \sum_\tau K_\tau. \quad (29)$$

For any  $n > 1$ , only transitions from a labor force size of  $n$  to  $n - 1$  with frequency  $\varphi(q)n$ , to  $n + 1$  with frequency  $\eta(\theta)v_n(q, n^d(q))$ , or to  $n = 0$  with frequency  $\delta_0$  can occur. Since the fraction of employers with  $n$  workers must equate flows into and out of the state, the steady state distribution over  $n$ , denoted  $P_0, P_1, \dots, P_n, \dots$  must satisfy

$$[\eta(\theta)\rho(q)v_n(q, Y/AKq) + \varphi(q)n + \delta_0] P_n \quad (30)$$

$$= \eta(\theta)\rho(q)v_{n-1}(q, Y/AKq)P_{n-1} + \varphi(q)(n+1)P_{n+1}, n \geq 1 \quad (31)$$

for any pair  $(q, Y/K)$  where, of course,  $n^d(q) = Y/AKq$ . Given that  $v_n(q, n^d(q)) = 0$  for all  $n \geq \bar{n}(q)$  and

$$\sum_0^\infty P_n = 1, \quad (32)$$

a unique solution to this second order difference equation exists which is continuous in all parameters.

Finally, market tightness, the ratio of the aggregate recruiting effort to aggregate search effort by definition, is

$$\theta = \frac{\sum_{\tau}^T AK_{\tau} \int_1^{\bar{q}} \sum_{n=0}^{\infty} v_n(x, Y/AKx) P_n(x, Y/AK) d\Gamma_{\tau}(x)}{\left( (1-e)s(R) + e \int_1^{\bar{q}} s(w(q)) dG(q) \right) L}. \quad (33)$$

## 5.2 Definition and Existence

**Definition:** A *steady state equilibrium* is a wage contingent search strategy  $s(w)$ , a product line recruiting effort strategy, value function, and employment size probability,  $v_n(q, Y/AKq)$ ,  $V_n(q, Y/AKq)$ , and  $P_n(q, Y/K)$  for all  $q \in [1, \bar{q}]$  and  $n \in \{0, 1, 2, \dots\}$ , an innovation rate,  $\gamma_{\tau}$ , and a measure of products supplied,  $K_{\tau}$ , for each firm type  $\tau \in \{1, 2, \dots, T\}$ , a firm entry innovation rate  $v$ , employment rate  $e$ , an aggregate final product flow  $Y$ , a productivity distribution over employment  $G(q)$ , a wage offer distribution function  $F(w)$ , and a labor market tightness  $\theta$  that satisfy equations (9), (17), (16), (30), (??), (32), (21), (22), (29), (25), (24), (??), (26), (28), and (33).

The first two of the following set of assumptions are standard regularity conditions and the third simplifies the existence proof without any essential loss of economic content.

**Assumption 1:** The matching function,  $\lambda(\theta) = \theta\eta(\theta)$ , is positive, continuous, increasing and concave, and  $\lim_{\theta \rightarrow 0} \lambda(\theta) = \lim_{\theta \rightarrow \infty} \eta(\theta) = 0$ .

**Assumption 2:** The cost functions,  $c_i(\gamma)$ ,  $i \in \{s, v, \gamma\}$ , are all continuous, strictly convex, and satisfy  $c_i(0) = c'_i(0) = 0$ .

**Assumption 3:** The number of firm types is finite, and the union of the supports of the distributions  $\Gamma_{\tau}(q)$ ,  $\tau = 1, 2, \dots, T$ , is discrete and contains a finite number of points.

**Proposition 6** *A steady state equilibrium exists.*

**Proof.** We use a fixed point argument to show that the assumptions guarantee the existence of at least one equilibrium in which  $0 < \theta < \infty$ . Denote the set of discrete productivity values as  $\{q_1, q_2, \dots, q_Q\}$  where without loss of generality  $q_{i+1} > q_i \geq 1$  so that  $q_Q = \bar{q}$  is the upper bound on the union of type contingent productivity distributions supports. Consider a vector  $\vec{y} = (y_0, y_1, \dots)$  where  $y_0 = \theta$ ,  $y_i = F(w(q_i))$ ,  $i = 1, 2, \dots, Q$ , and  $y_{Q+\tau} = K_{\tau}/K$ ,  $\tau = 1, 2, \dots, T$  and  $y_{Q+T+1} = Y$ . In other words, any  $y \in \Upsilon \equiv [\underline{\theta}, \bar{\theta}] \times [0, 1]^Q \times \Delta^T \times [0, \bar{Y}]$  specifies particular values for market tightness, the wage offer cumulative distribution function, and the distribution of product lines over firm types. Of course, total output  $Y$  is non-negative and bounded above by  $\bar{Y}$  as specified in equation (27). We wish to show that lower and upper bounds on  $\theta$ , denoted  $\underline{\theta}$  and  $\bar{\theta}$ , exists such that (i)  $0 < \underline{\theta} < \bar{\theta} < \infty$ , (ii) that the model define a continuous map from  $\Upsilon$  to itself, and (iii) that any fixed point of the map satisfies all the equilibrium conditions. Existence of an equilibrium candidate then follows by Brouwer's fixed point theorem.

From Assumptions 1 and 2, the search effort strategy vector,  $\vec{s}$  with typical element  $s_i = s(w)$ , as defined in by equations (9) exists and varies continuously with  $y$ . Given this

fact, equation (25) determines  $e$  as a continuous function of  $y$ . In turn, the wage equation (5) and steady state condition (26) determine the distribution of productivity c.d.f.  $G(q)$  as a vector of continuous functions of  $y$ . Similarly, the quit rate vector  $\vec{\varphi}$ , with representative element  $\varphi(w(q_i))$ , and the acceptance probability vector  $\vec{\rho}$ , with representative element  $\rho(w(q_i))$ , defined in equations (11) and (12) are continuous functions of  $y$ . These facts, the fact that  $\eta(\theta) = \lambda(\theta)/\theta$ , and  $n^d(q) = Y/AKq$  imply that the unique value function  $V_n(q, n^d(q))$  that solves equation (16), the optimal recruiting strategy defined by (20) and (17), and the employment size probabilities that solve (??) and (??) for each  $q$  are continuous functions of  $y$  and  $Y/K$ . By implication, the optimal R&D investment strategy vector  $\vec{\gamma} = (\gamma_1, \dots, \gamma_T)$  as defined by equation (23) is also continuous in  $y$  and  $Y/K$ .

Denote the optimal innovation strategy as a real value vector  $\vec{\gamma} : \Upsilon \times [0, \kappa] \rightarrow [0, \delta_0]^T$  where the representative element is the creation rate of a type  $\tau$  firm,  $\gamma_\tau$ , and the second argument of the function is  $Y/K$ . The upper bound  $\kappa$ , which is defined as the value of  $Y/K$  for which  $\max_\tau \{\gamma_\tau\} = \delta_0$ , exists by virtue of equation (23) and the fact that  $V_q(0, Y/AKq)$  tends to zero as  $Y/K$  tend to zero. As the first order conditions for optimal entry as defined by (24) imply that

$$c'_\gamma \left( \frac{v}{m} \right) = \sum_\tau c'_\gamma(\gamma_\tau) \phi_\tau$$

the entry rate,  $v$ , is also a continuous function of  $y$  and  $Y/K$ . Because  $\gamma_\tau$  for all  $\tau$  and therefore  $v$  are all decreasing in  $Y/K$ , equation (29) and the identity

$$K \equiv \sum_{\tau=1}^T K_\tau = \sum_{\tau=1}^T \frac{v(y, Y/K) \phi_\tau}{\delta_0 - \gamma_\tau(y, Y/K)} \quad (34)$$

uniquely determine  $K$  for any  $y \in \Upsilon$  where  $\gamma_\tau(y, Y/K)$  for all  $\tau$  and  $v(y, Y/K)$  are the innovation rates and entry rate functions referred to above. Since the right hand side of this expression is either positive or tends to infinity as  $Y/K$  tends to  $\kappa$  from below and is positive, continuous and strictly decreasing for all  $Y/K < \kappa$ , equation (34) implicitly defines  $K$  as a continuous function of  $y$  which is bounded above by  $Y/\kappa \leq \bar{Y}/\kappa$

By substituting back into equation (29), we obtain a non-negative product distribution vector function  $\vec{K} = (K_1, \dots, K_T)$  implicitly defined as a continuous function  $\vec{K} : \Upsilon \rightarrow R_+^T$  by the steady state conditions

$$K_\tau(y) = \frac{v(y, Y/K(y)) \phi_\tau}{\delta_0 - \gamma_\tau(y, Y/K(y))}, \tau \in \{1, 2, \dots, T\}.$$

Similarly, as equation (??) can be written as

$$Y = AL \left( 1 - e + e \sum_\tau K_\tau \int_1^{\bar{q}} \left( \sum_{n=0}^{Y/AKx} xn P_n(x) + x [Y/AKx - \hat{n}^d(x)] P_{\hat{n}^d+1}(x) \right) d\Gamma_\tau(x), \right)$$

where  $\hat{n}(q)$  is the largest integer less than or equal to  $n^d(q) = Y/AKq$ , it is a continuous function of  $y$  with bounded range  $[0, \bar{Y}]$ . Obviously, equations (28) can be written as

$$F(w(q_i)) = \frac{\sum_\tau K_\tau(y) \int_1^q \sum_{n=0}^\infty v_n(x, Y/AK(y)x) P_n(x, Y/AK(y)x) d\Gamma_\tau(x)}{\sum_\tau K_\tau(y) \int_1^{\bar{q}} \sum_{n=0}^\infty v_n(w, Y/AK(y)x) P_n(w, Y/AK(y)x) d\Gamma_\tau(w)}, i = 1, 2, \dots, Q,$$

which defines a continuous function  $\vec{F} : \Upsilon \rightarrow [0, 1]^Q$  that represents the wage offer c.d.f. Finally, as the definition of market tightness given in equation (33) can be written as

$$\theta = \frac{\sum_{\tau} K_{\tau} \int_1^{\bar{q}} \sum_{n=0}^{\infty} v_n(x, Y/AK(y)x) P_n(x, Y/K(y)) d\Gamma_{\tau}(x)}{\left( hs(R) + e \int_1^{\bar{q}} s(w(q)) dG(q) \right) L}$$

it too is a continuous function defined on the set of real vectors  $\Upsilon$ .

To this point, we have shown that the equilibrium conditions define a continuous vector function  $\vec{y} : \Upsilon \rightarrow \Upsilon = [0, \infty] \times [0, 1]^Q \times \Delta^T \times [0, \bar{Y}]$ . Furthermore, the assumption that  $\lim_{\theta \rightarrow \infty} \eta(\theta) = 0$  implies that  $v(w, Y/AKq) \rightarrow 0$  as  $\theta \rightarrow 0$  for any  $w$  and the other components of  $y$  from equation (17), and, consequently,  $\theta \rightarrow 0$  as defined by equation (??). In short,  $\theta = 0$  is not a fixed point of the map. Similarly, the assumption that  $\lim_{\theta \rightarrow 0} \lambda(\theta) = 0$  implies that  $s(w) \rightarrow 0$  from equation (9), and, consequently,  $\theta \rightarrow \infty$  rules out  $\infty$  as a fixed point of the map. Hence, by continuity at least one real positive solution to (??) for  $\theta$  where  $y_0 = \theta$  exists for any choice of the truncated vector  $y^* \equiv (y_1, \dots, y_{Q+T})$  which is bounded away from zero and infinity. Specifically, if one lets  $\Theta(y^*)$  define the solution set associated with any choice of  $y^* \in [0, 1]^Q \times \Delta^T \times [0, z]$ , we know that  $\vec{y} : \Upsilon \rightarrow \Upsilon$  where

$$\underline{\theta} = \min_{y^* \in [0, 1]^Q \times \Delta^T} \Theta(y^*) > 0$$

and

$$\bar{\theta} = \max_{y^* \in [0, 1]^Q \times \Delta^T} \Theta(y^*) < \infty.$$

Hence, existence of at least one equilibrium solution exists by Brouwer's fixed point theorem. ■

## 6 Simulation Experiments

To be completed.

## 7 Summary

To be completed.

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