A Microfoundation for Production Functions: Assignment of Heterogenous Workers to Heterogenous Jobs*

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Abstract

The assignment model proposed in this paper provides a microeconomic foundation for aggregate production functions. In this model, the shape of the production function crucially depends on the distribution of workers and jobs and the type of technological changes crucially depends on the evolution of these distributions. If jobs are Beta distributed, and the product of a worker-job match is Cobb-Douglas and function of the output level, then the aggregate production function is shown to be Constant Ratio of Elasticities of Substitution. If in addition the Beta distribution is symmetric, the aggregate production function degenerates to the Constant Elasticity of Substitution with general returns to scale. These results provide a way to evaluate how stringent assumptions about the type of production functions or technological change are by comparing the implied distribution of jobs and its evolution over time to observations of the distribution of jobs and its evolution over time.

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1 Introduction

In very different fields of economics, economic inference and policy evaluation require economists to parametrize a production function that links measures of input factors to measures of output. While doing so, strong assumptions are implicitly made about microeconomic variables such as the distribution of factors (see Houthakker (1955-1956)), the distribution of ideas (see Jones (2005)), the distribution of productivity (see Rosen (1978)) or the distribution of jobs (this paper). For instance, in the literature about skill-biased technology change and rising skill-premium, a standard assumption is that aggregate production is of the Constant Elasticity of Substitution (CES) type and technical change is skilled-labor augmenting.[1] What the CES assumption implies for the distribution of jobs or what the skilled-labor augmenting technical change implies for the evolution of the distribution of jobs over time is not known. A general drawback of using production functions is that these functions lack micro-economic foundations that would enable us to justify our parametrization. Without identifying the microeconomic forces that govern the structure of production, economic inference and policy implications based on reduced form aggregate production functions may be highly hazardous.

Surprisingly enough, very little is known about the microeconomic foundations of production functions. To my knowledge, only Houthakker (1955-1956), Levhari (1968), Rosen (1978) and more recently Jones (2005) and Lagos (2007) have viewed production functions as reduced form of micro-founded models. In Houthakker's (1955-1956) original work, the aggregate production function is derived from the distribution of inputs across productive cells (e.g. firms, machines or workers). Houthakker shows that when the distribution of inputs is of the generalized Pareto type, the aggregate production function then has the Cobb-Douglas form.[2] A shortcoming of this approach is that inputs are randomly distributed across productive cells whereas inputs mix are more likely to be the result of a matching process

[1] See Katz and Murphy (1992) and Acemoglu (2002) and references therein.

[2] Levhari (1968) showed that this result could be extended to CES production function when inputs are distributed according to a Beta distribution. Jones (2005) builds on this model and derives the shape of the production function and the direction of technical change from the distribution of ideas. Lagos (2007) also builds on Houthakker's model and derive the shape of the aggregate production function in a model with search frictions in the labor market.

at the level of the productive cell. Rosen's (1978) tasks assignment model accounts for the non random assignment of inputs. In his model, the problem is to assign workers of different productive types to heterogenous tasks so as to maximize aggregate output. Rosen (1978) then derives the shape of the aggregate production function from the assignment of workers to tasks. When tasks are uniformly distributed and under additional assumptions on the productivity of workers in the various tasks, the aggregate production function is of the CES type with constant returns to scale.

In this paper, I consider a tasks assignment model with two types of workers and heterogeneity of workers within types and, general distributional form of tasks. In contrast to Rosen's (1978) model that ignores capital, each task is associated with a unit of capital, a machine for the sake of the argument, and output is therefore produced using both capital and labor inputs. The assignment problem is solved using Ricardo's principles of comparative advantage and differential rents. In the economy considered, each machine can only be operated by one worker at a time. As a result, the shape of the aggregate production function is governed principally by the distribution of tasks in contrast with Rosen's (1978) model in which the shape of the production function is driven by the productivity of the worker-task matches. This makes the model of this paper very convenient to check the validity of assumptions made when using macro production functions. While information on the physical productivity of each worker-task pair is rarely available in data sets, information about the distribution of jobs can readily be derived in most data sets using occupational codes for instance.

Closed form solutions for the shape of the aggregate production function can be recovered in this general assignment model. In particular, I will show that the aggregate production function has the Constant Ratio of Elasticities of Substitution (CRES) type when tasks are distributed according to the Beta distribution and the productivity of a match worker-task is Cobb-Douglas and function of the output level. With the additional restriction that the Beta distribution is symmetric, the aggregate production function degenerates to the CES type with general returns to scale.

The remaining structure of the paper is as follows. In the next section, I present the model of tasks assignment with multidimensional skills. In section 3, a parametrization of the model is proposed so that the aggregate production function is of the CRES type. This parametrization enables us to reveal what microeconomic assumptions on the distribution of tasks are made when imposing the aggregate production function to be of the CES type. In section 4, using the results of the model, I explain that the type of changes in the distribution of tasks implicitly imposed in the skill-biased technical change literature (see Katz and Murphy (1992) or Acemoglu (2002)), by using a CES production function with skill-labor augmenting technical change, does *not* fit well with recent empirical evidence about job polarization (see Autor et al. (2006)) and organizational change (see Brynjolfsson (1995), Bresnahan (1999) and Bresnahan et al. (2002) and van Reenen and Caroli (2001) for instance).

2 Tasks assignment model with two types of heterogenous workers and a continuum of tasks

2.1 Setting of the model

Consider an economy producing a composite commodity by means of the input of an infinite number of different tasks. Each task is associated with a unit of capital, a machine for the sake of the argument, and the various tasks correspond to machines with different characteristics.[3] In this economy, output Y is obtained by summing up the production in each single task v from the continuum $v \in (0, 1)$. There is therefore perfect substitution among the output of each task but each machine can only be operated by one worker at a time.[4] The distribution of tasks is exogenous in the sense that, in the period under consideration, the density distribution of tasks does not depend on wages. The distribution of tasks has probability density function $d(v)$ and the cumulative density function $F(v^*) = \int_0^{v^*} d(v)dv$.[5] Moreover, the economy is perfectly competitive so that no worker and no owner of capital can affect

[3] This part of the model is to a large extent similar to the differential rents models described in Sattinger (1979 and 1993). The terminology "task" and "machine" are interchangeable throughout the paper. In general I will use "task" for the sake of simplicity but when needed I will refer explicitly to machines.

[4] This contrasts with Rosen's (1978) model that ignores capital. In Rosen's model, tasks from different jobs are needed in fixed proportions to produce aggregate output, there is no substitution possibilities between tasks, but each job can be filled with more than one worker.

[5] This economy is assumed to be large enough so that the density distribution of both workers and tasks are continuous.

the wage and rental rates.

Workers are heterogenous in terms of their type and level of skills. For simplicity there are two types of skills, say type 1 and type 2. The distribution of workers of each type of skills is exogenous and given by the density distribution $s_k(t_k)$ for $k = 1, 2$ with $t_k \in [\underline{t_k}, \overline{t_k}]$. Output at task v can be produced by workers with different types and levels of skills. However, workers of different types and levels of skills differ in their productivity. Workers supplying t_k units of skills of type k can produce $p_k(t_k, v)$ units of output when assigned to machine v. Without loss of generality, I assume that workers supplying skills of type 1 have a comparative advantage in tasks v close to 0 and workers supplying skills of type 2 have a comparative advantage in tasks v close to 1. I assume further that productivity increases with the level of skills supplied. Moreover, among workers supplying skills of type 1, those with higher t_1 skills are more productive in tasks v close to 0 and among workers supplying skills of type 2, those workers with higher t_2 skills are more productive in tasks v close to 1.

Assumption A: i) Comparative advantage of skills types, i.e. $\frac{\partial p_1(t_1,v)}{\partial v} < 0$ and $\frac{\partial p_2(t_2,v)}{\partial v} > 0$ $\forall v, t_k$, *ii)* absolute advantage of skilled workers, $\frac{\partial p_k(t_k,v)}{\partial t_k} > 0$ $\forall v, t_k$, *iii)* hierarchical productivity $\frac{\partial^2 p_1(t_1,v)}{\partial t_1 \partial v} \leq 0$ and $\frac{\partial^2 p_2(t_2,v)}{\partial t_2 \partial v} \geq 0$ $\forall v, t_k$.

2.2 Tasks assignment and equilibrium

Following Sattinger (1979 and 1993), the general equilibrium of this model is derived in three steps once we assume that in the period under consideration the density distribution of tasks does not depend on wages. In the first step, we make a tentative assumption about the assignment of workers to tasks in equilibrium. The second step consists to derive the associated equilibrium wages for this assignment. Finally, in the third step, we check whether the second order conditions for equilibrium are satisfied by the equilibrium wages derived in step 2.

Step 1: *Tentative tasks assignment*

Given assumption A $i)$, an efficient assignment of workers to tasks will maximize output by assigning workers supplying skills of type 1 to tasks $(0, \varepsilon)$ and workers supplying skills of type 2 to tasks $(\varepsilon, 1)$ where ε is the marginal task in equilibrium.[6] Given assumption *A ii)* and *iii)*, among workers supplying skills of type 1, those with the highest level of skill 1 will be assigned to task 0 and so on until the marginal task ε is assigned to those workers supplying the lowest level of skill 1.[7] By symmetry, the tasks $(\varepsilon, 1)$ are assigned to workers supplying skills of type 2. Workers supplying the lowest level of skills of type 2 are assigned to task ε and so on until those workers with the highest level of skills 2 are assigned to task 1.

This efficient tasks assignment results in a mapping function v_1 that associates to each value of skills t_1 a single value of task $v \in (0, \varepsilon)$, i.e. $v_1 = v_1(t_1)$ with $v_1' < 0$, and a mapping function v_2 that associates to each value of skills t_2 a single value of task $v \in (\varepsilon, 1)$, i.e. $v_2 = v_2(t_2)$ and $v_2' > 0$.[8] The functions v_1 and v_2 are monotonic, decreasing and increasing respectively on $t_1 \in (\underline{t_1}, \overline{t_1})$ and $t_2 \in (\underline{t_2}, \overline{t_2})$ since within types of skills more skilled workers are assigned to more productive machines in equilibrium.

To show how the assignment equilibrates the supply of and demand for skills, I split the interval of tasks assigned to type 1 workers, $v \in (0, \varepsilon)$ into N intervals of equal length $\Delta_i = \Delta = \frac{\varepsilon}{N}$ for $i = 1, ..., N$. Labor demand in interval i, that is the number of tasks to be filled in interval i, is $\int_{(i-1)\Delta}^{i\Delta} d(v)dv$. To equilibrate each tasks interval firms will assign the $\int_0^{\Delta} d(v)dv$ most skilled workers to the first interval, the following $\int_{\Delta}^{2\Delta} d(v)dv$ most skilled workers to the second interval and so on and so forth until the last interval is filled with the $\int_{(N-1)\Delta}^{N\Delta} d(v)dv$ least skilled workers.[9]

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[6] Firms are indifferent between assigning a worker with type 1 or type 2 at task ε.

[7] This hierarchical sorting within education resembles the differential rents models proposed by Sattinger (1979) and (1993).

[8] See Sattinger (1975). Functions v_i, $i = 1, 2$ play the same role as the function $h(g)$ in Sattinger (1975) p. 356, where g is workers' ability (single scale) and $h(g)$ the difficulty (single scale) of the task performed by workers with ability g in equilibrium and, $c(u)$ in Teulings (1995a), (1995b) and (2005) where u is the normalized level of skills and $c(u)$ the associated job complexity in equilibrium.

[9] Note however that, since the density distribution of workers by level of skills needs not correspond to the density distribution of tasks, the skill differences between the most and least skilled workers in each interval needs not be the same. For instance, suppose that skills are normally distributed among workers of type 1 and tasks are uniformly distributed on $(0, 1)$. The skill differential will first decrease (upper tail of a normal distribution is thinner than the upper tail of a uniform distribution) and increase once the median skilled worker has been assigned. To put it in Tinbergen's terms "the supply distribution of skills has to be deformed so as to coincide with the demand distribution otherwise there will not be an equilibrium." The assignment of tasks to workers deforms (stretches) the

The density of workers' skills can therefore directly be derived from the density of tasks by performing the transformation of variables $v = v_k(t_k)$ and noting that $dv = v_k'(t_k)dt_k$. This yields:

$$\int_{v_1(\underline{t_1})}^{v_1(\overline{t_1})} d(v_1(t_1))v_1'(t_1)dt_1 = \int_{\underline{t_1}}^{\overline{t_1}} s_1(t_1)dt_1 \text{ for } v_1(\underline{t_1}) \leq \varepsilon \tag{1}$$

$$\int_{v_2(\underline{t_2})}^{v_2(\overline{t_2})} d(v_2(t_2))v_2'(t_2)dt_2 = \int_{\underline{t_2}}^{\overline{t_2}} s_2(t_2)dt_2 \text{ for } v_2(\underline{t_2}) \geq \varepsilon \tag{2}$$

The density of individuals with skills of type k is therefore $s_k(t_k) = d(v_k(t_k))v_k'(t_k)$. These are first order nonlinear nonautonomous differential equations.[10]

Step 2: *Equilibrium wages*

The owner of machine v seeks to maximize the profits derived from its machine. The profits from assigning a worker with skills t_k are $p_k(t_k, v) - w_k(t_k)$. The owner will therefore compare the productivity increase to the wage increase associated to a worker with higher skills t_k for all k. This yields the following first order condition:

$$\frac{\partial p_k(t_k, v)}{\partial t_k} = w_k'(t_k) \ \forall k = 1, 2 \tag{3}$$

Note that from assumptions *A ii*), we therefore have $w_k'(t_k) > 0 \ \forall k = 1, 2$. Moreover, the owners of machine ε are indifferent between employing the worker supplying the lowest level of skills of type 1, $t_{1,\varepsilon}$, or the worker supplying the lowest level of skills of type 2, $t_{2,\varepsilon}$. Stated otherwise, the rents of the owners of machines ε are equal whether worker $t_{1,\varepsilon}$ or $t_{2,\varepsilon}$ are assigned to machine ε: $p_1(t_{1,\varepsilon}, \varepsilon) - w_1(t_{1,\varepsilon}) = p_2(t_{2,\varepsilon}, \varepsilon) - w_2(t_{2,\varepsilon})$.

Equation 3 gives the wage differential at task v. This wage differential does not hold for values of t_k other than $t_k = v_k^{-1}(v)$ and therefore depends

density distribution of skills of each type such as to make it fit the distribution of tasks in equilibrium.

[10] Examples of closed form solutions are available in the one skill dimension case in Sattinger (1975), (1979) and (1993). Teulings (1995a), (1995b) and (2005) solves a second order differential equation in the one skill scale case.

on the equilibrium assignment. Evaluating the differential equation 3 for $v = v_k(t_k)$ and integrating over t_k yields the wage functions for workers supplying skills of type k.

Step 3: *Second order conditions*

The equilibrium assignment defined by the mapping functions v_k, is a valid one only when the firm's second order condition to profits maximization, that is profits are concave in t_k, is satisfied. Put in equation:

$$\begin{aligned} \left[\frac{\partial^2 p_k(t_k, v)}{\partial t_k^2}\right]_{v=v_k(t_k)} - w_k''(t_k) &< 0 \ \forall k = 1, 2 \\ &\Leftrightarrow \\ -\left[\frac{\partial^2 p_k(t_k, v)}{\partial t_k \partial v} v_k'\right]_{v=v_k(t_k)} &< 0 \end{aligned}$$

since $w_k''(t_k) = \left[\frac{\partial^2 p_k(t_k,v)}{\partial t_k^2}\right]_{v=v_k(t_k)} + \left[\frac{\partial^2 p_k(t_k,v)}{\partial t_k \partial v} v_k'(t_k)\right]_{v=v_k(t_k)}$.

Since $v_1' < 0$ and $v_2' > 0$ this second order condition therefore implies that $\left[\frac{\partial^2 p_1(t_1,v)}{\partial t_1 \partial v}\right]_{v=v_1(t_1)} < 0$ and $\left[\frac{\partial^2 p_2(t_2,v)}{\partial t_2 \partial v}\right]_{v=v_2(t_2)} > 0$. Hence, as long as the cross derivative $\frac{\partial^2 p_1(t_1,v)}{\partial t_1 \partial v}$ is negative and the cross derivative $\frac{\partial^2 p_2(t_2,v)}{\partial t_2 \partial v}$ is positive, that is as long as assumption *A*2 *iii*) holds, an assignment where within educational groups more skilled workers get more productive machines, i.e. $v_1' < 0$ and $v_2' > 0$, is valid.

2.3 The shape of the aggregate production function

Using the function v_k, aggregate output level is given by:

$$Y = \int_0^{\varepsilon} p_1(v_1^{-1}(v), v) d(v) dv + \int_{\varepsilon}^{1} p_2(v_2^{-1}(v), v) d(v) dv$$

Moreover, the demand for type k workers per unit of output is directly derived by integrating the number of workers per unit of output for each skills type (the inverse of the productivity of workers) over the spectrum of tasks. The demand for type k workers per unit of output is then given by:

$$\frac{D_1(\varepsilon)}{Y} = \int_0^{\varepsilon} \frac{d(v)}{p_1(v_1^{-1}(v), v)} dv \tag{4}$$

$$\frac{D_2(\varepsilon)}{Y} = \int_{\varepsilon}^{1} \frac{d(v)}{p_2(v_2^{-1}(v), v)} dv \tag{5}$$

Solving for the marginal task ε, one can derive, under invertible conditions, the shape of the production function linking the output level Y to the employment level of type 1 and 2 workers, i.e. $Y = h(D_1, D_2)$. In the general case, analytical solutions for h will not be possible. However, as I will show in the next section, under certain restrictions, h could take the general form of a CRES production function which admits the CES production function as a special case.

Nevertheless, the elasticity of substitution between both types of workers can be derived formally as:

$$\sigma(\varepsilon) \equiv \frac{d \ln \left(\frac{D_1/Y}{D_2/Y}\right)}{d \ln MRS_{12}(\varepsilon)} = -\frac{\frac{d(\varepsilon)/p_1(v_1^{-1}(\varepsilon),\varepsilon)}{D_1(\varepsilon)/Y} + \frac{d(\varepsilon)/p_2(v_2^{-1}(\varepsilon),\varepsilon)}{D_2(\varepsilon)/Y}}{\frac{p_1'(v_1^{-1}(\varepsilon),\varepsilon)}{p_1(v_1^{-1}(\varepsilon),\varepsilon)} - \frac{p_2'(v_2^{-1}(\varepsilon),\varepsilon)}{p_2(v_2^{-1}(\varepsilon),\varepsilon)}} \tag{6}$$

where $MRS_{12}(\varepsilon) \equiv \frac{d(D_1(\varepsilon)/Y)/d\varepsilon}{d(D_2(\varepsilon)/Y)/d\varepsilon} = -\frac{p_2(v_2^{-1}(v),v)}{p_1(v_1^{-1}(v),v)}$. And, MRS_{ij} stands for Marginal Rate of Substitution between input i and j.

From this, we can see that the elasticity of substitution between both types of workers depends crucially on $d(v)$ the distribution of tasks, the mapping functions v_j and therefore the distribution of skills, and $p_k(t_k, v)$ the productivity of a worker-task match.

3 Parametrization[11]

3.1 Tasks distribution

Assume that the distribution of tasks follows a Beta distribution on $(0,1)$ and has density function $d(v|d_1,d_2) = Av^{d_1}(1-v)^{d_2}$ with $d_j \geq 0$ and $A = 1/B(d_1+1, d_2+1)$ and $B(.)$ is the Beta function and cumulative distribution $F(v^*|d_1,d_2) = \Pr(v < v^*) = \int_0^{v^*} d(v|d_1,d_2)dv$. The mean task is given by $E[v] = \frac{d_1+1}{2+d_1+d_2}$ ($E[v] = \frac{1}{2}$ when $d_1 = d_2$) and the variance by $V[v] = \frac{(d_1+1)(d_2+1)}{(2+d_1+d_2)(3+d_1+d_2)}$, with $\frac{\partial V[v]}{\partial d_k} < 0$. Moreover, the distribution is skewed toward 0 when $d_1 > d_2$ and vice versa.

The Beta distribution is appealing because its support ranges from 0 to 1, it has only two parameters, and its shape is extremely flexible. If $d_1 > 0$ and $d_2 > 0$ the distribution is unimodal. If $d_1 = d_2 = d$ and $d = 0$ tasks are uniformly distributed (as in Rosen (1978)). Moreover, for $d > 1$ the Beta distribution and the normal distribution with average $\frac{1}{2}$ and variance equal to $\frac{(d_1+1)(d_2+1)}{(2+d_1+d_2)(3+d_1+d_2)}$ look alike.

3.2 Skills distributions and mapping functions

The mapping functions v_k equilibrate supply and demand of skills of both types so that $s_k(t_k) = d(v_k(t_k))v_k'(t_k)$. The mapping functions are therefore the solution of these first order nonlinear nonautonomous differential equations. This type of differential equations generally do not give rise to closed form solutions.

Hence, further assumptions are required to solve the problem. Rather than impose the shape of the density distribution of skills of each type and solve the differential equations for the mapping functions, I impose the shape of the mapping functions and solve $s_k(t_k) = d(v_k(t_k))v_k'(t_k)$ for the density functions s_k. Suppose that the function assigning skills of type 1 to tasks in equilibrium is such that $v = v_1(t_1) = 1 - \left(\frac{\underline{t}_1}{t_1}\right)^{1/a_1}$ with $a_1 \geq 0$ to satisfy $v_1' < 0$ and the function assigning skills of type 2 to tasks is such that $v = v_2(t_2) = \left(\frac{\underline{t}_2}{t_2}\right)^{1/a_2}$ with $a_2 \geq 0$ to satisfy $v_2' > 0$. Note that the parameters

[11] Since the primary interest in this paper is the shape of the aggregate production function, the parametric shape of the equilibrium wage functions is not discussed here. These results are available, though, upon request to the author.

a_k indicate the percentage increase in the type 1 (respectively type 2) skills of workers assigned to a task situated 1 percent more to the left in equilibrium (respectively to the right), i.e. $\frac{\partial \ln t_1}{\partial \ln(1-v)} = -a_1$ and $\frac{\partial \ln t_2}{\partial \ln v} = a_2$.[12]

The density of workers with t_1 skills of type 1 is given by:

$$s_1(t_1) = \frac{A}{a_1\overline{t_1}}\left(1-\left(\frac{t_1}{\overline{t_1}}\right)^{1/a_1}\right)^{d_1}\left(\frac{t_1}{\overline{t_1}}\right)^{d_2/a_1+1/a_1-1} \tag{7}$$

for $0 < \underline{t_1} < t_1 \leq \overline{t_1}$.

The density of workers with t_2 skills of type 2 is given by:

$$s_2(t_2) = \frac{A}{a_2\overline{t_2}}\left(1-\left(\frac{t_2}{\overline{t_2}}\right)^{1/a_2}\right)^{d_2}\left(\frac{t_2}{\overline{t_2}}\right)^{d_1/a_2+1/a_2-1} \tag{8}$$

for $0 < \underline{t_2} < t_2 \leq \overline{t_2}$.

The total supply of type 1 and type 2 workers is equal to the density of tasks between 0 and ε and the density of tasks between ε and 1 respectively.

$$S_1 = \int_{\underline{t_1}}^{\overline{t_1}} s_1(t_1)dt_1 \equiv \int_0^{\varepsilon} d(v)dv = F(\varepsilon|d_1,d_2) = D_1(\varepsilon) \tag{9}$$

$$S_2 = \int_{\underline{t_2}}^{\overline{t_2}} s_2(t_2)dt_2 \equiv \int_{\varepsilon}^{1} d(v)dv = 1-F(\varepsilon|d_1,d_2) = D_2(\varepsilon) \tag{10}$$

3.3 Productivity of a match worker-task

Suppose further that the productivity of workers with skills k at task v is Cobb-Douglas as follows:

$$p_1(t_1,v) = b_1 t_1^{m_1}(1-v)^{n_1} \tag{11}$$

$$p_2(t_2,v) = b_2 t_2^{m_2} v^{n_2} \tag{12}$$

where $b_k > 0$ are parameters indicating the efficiency units of workers with skills k. m_k indicates the elasticity of output with respect to skills, with m_k

[12]Note that for $a_1 = a_2 = 0$, workers with education k have homogenous skills $t_k = \overline{t_k}$. Therefore, for $a_1 = a_2 = 0$, the model reduces to Rosen's (1978) task assignment model.

> 0 to satisfy assumption A $ii)$, i.e. $\partial p_k/\partial t_k > 0$. n_k indicates the elasticity of output with respect to tasks, with $n_k > 0$ to satisfy assumptions A $i)$ and $iii)$.

Roughly speaking, n_j indicates the degree of heterogeneity of machines in terms of productivity[13] whereas m_j indicates the degree of heterogeneity of workers in terms of productivity.[14]

The aggregate output level is obtained by summing up the product of each worker or equivalently the product at each task in equilibrium. This gives:

$$\begin{aligned} Y &= \int_0^{\varepsilon} p_1(v_1^{-1}(v), v)d(v)dv + \int_{\varepsilon}^{1} p_2(v_2^{-1}(v), v)d(v)dv \\ &= b_1\overline{t_1}^{m_1} A \int_0^{\varepsilon} v^{d_1}(1-v)^{2d_2}dv + \\ & \quad b_2\overline{t_2}^{m_2} A \int_{\varepsilon}^{1} v^{2d_1}(1-v)^{d_2}dv \end{aligned}$$

After recognizing that $A_1 v^{d_1}(1-v)^{2d_2}dv$ is the density of the Beta distribution with parameter d_1 and $2d_2$ with $A_1 = \frac{1}{B(d_1,2d_2)}$ and $A_2 v^{2d_1}(1-v)^{d_2}$ is the density of the Beta distribution with parameters $2d_1$ and d_2 with $A_2 = \frac{1}{B(2d_1,d_2)}$, the aggregate output level reads as:

$$Y = b_1\overline{t_1}^{m_1}\frac{A}{A_1}F(\varepsilon|d_1, 2d_2) + b_2\overline{t_2}^{m_2}\frac{A}{A_2}(1 - F(\varepsilon|2d_1, d_2))$$

3.4 The shape of the aggregate production function

Given the structural form in equation 11 and 12, the employment of type k workers per unit of output in equilibrium reads as:

[13] As $n_j \to 0$ all tasks tend to be equally productive.

[14] As $m_j \to 0$ workers with different levels of type j skills tend to be equally productive.

$$\frac{D_1(\varepsilon)}{Y} = \int_0^{\varepsilon} \frac{d(v|d_1,d_2)}{p_1(v_1^{-1}(v),v)}dv = A\int_0^{\varepsilon} \frac{1}{b_1\overline{t_1}^{m_1}} \frac{v^{d_1}(1-v)^{d_2}}{(1-v)^{m_1a_1+n_1}}dv \quad (13)$$

$$\frac{D_2(\varepsilon)}{Y} = \int_{\varepsilon}^1 \frac{d(v|d_1,d_2)}{p_2(v_2^{-1}(v);v)}dv = A\int_{\varepsilon}^1 \frac{1}{b_2\overline{t_2}^{m_2}} \frac{v^{d_1}(1-v)^{d_2}}{v^{m_2a_2+n_2}}dv \quad (14)$$

General analytical solutions exist for these integrals. An interesting special case of which is met when $d_2 = m_1a_1 + n_1$ and $d_1 = m_2a_2 + n_2$. The solutions of these integrals are then:

$$\frac{D_1(\varepsilon)}{Y} = \frac{A}{b_1\overline{t_1}^{m_1}}\int_0^{\varepsilon} v^{d_1}dv = \frac{A}{b_1\overline{t_1}^{m_1}}\frac{1}{d_1+1}\varepsilon^{d_1+1} \quad (15)$$

$$\frac{D_2(\varepsilon)}{Y} = \frac{A}{b_2\overline{t_2}^{m_2}}\int_{\varepsilon}^1 (1-v)^{d_2}dv = \frac{A}{b_2\overline{t_2}^{m_2}}\frac{1}{d_2+1}(1-\varepsilon)^{d_2+1} \quad (16)$$

Note that, for a symmetric distribution of tasks, i.e. $d_1 = d_2 = \frac{1}{\theta} - 1$, solving the system for the marginal task ε yields:

$$\varepsilon = \left(\frac{D_1(\varepsilon)}{Y}\frac{b_1\overline{t_1}^{m_1}}{A}\frac{1}{\theta}\right)^{\theta} = 1 - \left(\frac{D_2(\varepsilon)}{Y}\frac{b_2\overline{t_2}^{m_2}}{A}\frac{1}{\theta}\right)^{\theta} \quad (17)$$

$$\Leftrightarrow$$

$$Y = \frac{1}{\theta A}\left[\left(b_1\overline{t_1}^{m_1}D_1(\varepsilon)\right)^{\theta} + \left(b_2\overline{t_2}^{m_2}D_2(\varepsilon)\right)^{\theta}\right]^{\frac{1}{\theta}} \quad (18)$$

Equation 18 reads as a CES production function with elasticity of substitution parameter $\sigma = \frac{1}{1-\theta}$. The parameter θ indicating the curvature of the relative productivity of workers in the various tasks in equilibrium and is related to the distribution of tasks.

Result R1: When i) the productivity of a match worker-task is Cobb-Douglas, i.e. as defined as in equations 11 and 12, ii) the mapping functions are given by $v = v_1(t_1) = 1-\left(\frac{t_1}{\overline{t_1}}\right)^{1/a_1}$ and $v = v_2(t_2) = \left(\frac{t_2}{\overline{t_2}}\right)^{1/a_2}$, iii) $d_k = n_j+m_ja_j$ and

iv) the distribution of tasks follows a symmetric Beta distribution, then the aggregate production function resulting from the assignment of multi-skilled heterogenous workers to heterogenous tasks has the CES shape.

Plugging the expressions of $d(v)$, $p_k(v_k^{-1}(v), v)$ and $D_k(\varepsilon)/Y$ into equation 6 yields the elasticity of substitution in the marginal task as:

$$\sigma(\varepsilon) = \frac{d_1 + 1}{d_1}$$

This result has two important implications. Implication 1 is that the elasticity of substitution is constant over the marginal task. Implication 2 is that the more concentrated the tasks around 0.5, i.e. the larger d_1, the smaller the elasticity of substitution and vice versa. In fact, for uniform distribution, the elasticity of substitution is infinitely large and the function of production is linear.

Result R1 is very convenient as it links the assignment literature to the empirical literature of wage inequality in which the CES production function has been extensively used. However, it is restrictive in the sense that only symmetric distributions of machines are covered, though these distributions can range from the uniform distribution when $\theta = 1$ ($\sigma \to \infty$ linear production function), inverted-U shape for $2 > \frac{1}{\theta} > 1$ ($\sigma \in (2, \infty)$) and normal look alike distributions for $\frac{1}{\theta} > 2$, ($\sigma \in (1, 2)$).

Nevertheless, one can generalize this finding and solve for more general shapes of the production function assuming that the productivity of workers in the various tasks depends on how many units of aggregate output are produced. Assume for instance that producing an extra unit of output affects the productivity of all workers in every tasks. Let $b_k \equiv b_k(Y) = r_k Y^{1-\frac{\theta}{\theta_k}}$ with $r_k > 0$, $\theta > 0$ and $\theta_k = \frac{1}{d_k+1}$, so that employing more workers in each task reduces (increases) the productivity per unit of output of each worker in every tasks, i.e. $\frac{\partial p_k}{\partial Y} < 0$ for $1 - \frac{\theta}{\theta_k} < 0$ (respectively $\frac{\partial p_k}{\partial Y} > 0$ for $1 - \frac{\theta}{\theta_k} > 0$).

Under this assumption, the demand for type k workers reads as:

$$\frac{D_1(\varepsilon)}{Y} = \frac{A}{r_1 Y^{1-\frac{\theta}{\theta_1}} \overline{t_1}^{m_1}} \frac{1}{d_1 + 1} \varepsilon^{d_1+1} \tag{19}$$

$$\frac{D_2(\varepsilon)}{Y} = \frac{A}{r_2 Y^{1-\frac{\theta}{\theta_2}} \overline{t_2}^{m_2}} \frac{1}{d_2 + 1} (1 - \varepsilon)^{d_2+1} \tag{20}$$

Which solving for the marginal task yields:

$$\begin{aligned}\varepsilon &= \frac{1}{Y^{\theta}}\left(D_1(\varepsilon)\frac{r_1\overline{t_1}^{m_1}}{\theta_1 A}\right)^{\theta_1} = 1-\frac{1}{Y^{\theta}}\left(D_2(\varepsilon)\frac{r_2\overline{t_2}^{m_2}}{\theta_2 A}\right)^{\theta_2}\\ &\Leftrightarrow\\ Y &= \left[\left(\frac{r_1\overline{t_1}^{m_1}}{\theta_1 A}D_1(\varepsilon)\right)^{\theta_1}+\left(\frac{r_2\overline{t_2}^{m_2}}{\theta_2 A}D_2(\varepsilon)\right)^{\theta_2}\right]^{\frac{1}{\theta}} \qquad (21)\end{aligned}$$

Equation 21 reads as the CRES production function (see Houthakker (1960) and Dick and Medoff (1975)). This function degenerates to a CES production function with returns to scale $\frac{\theta_1}{\theta}$ when $\theta_1 = \theta_2$ that is for symmetric tasks distributions. When $\theta_1 = \theta_2 = \theta$, the function degenerates to the constant returns to scale CES production function discussed above.

Result R2: When i) the productivity of a match worker-task is Cobb-Douglas and is function of the output level, i.e. as defined as in equations 11 and 12, ii) the mapping functions are given by $v = v_1(t_1) = 1 - \left(\frac{t_1}{\overline{t_1}}\right)^{1/a_1}$ and $v = v_2(t_2) = \left(\frac{t_2}{\overline{t_2}}\right)^{1/a_2}$, iii) $d_k = n_j + m_j a_j$, and iv) the distribution of tasks follows a Beta distribution, then the aggregate production function resulting from the assignment of multi-skilled heterogenous workers to heterogenous tasks is of the CRES shape.

Plugging the expressions of $d(v)$, $p_k(v_k^{-1}(v), v)$ and $D_k(\varepsilon)/Y$ into equation 6 yields the elasticity of substitution in the marginal task as:

$$\sigma(\varepsilon) = \frac{\frac{d_1+1}{\varepsilon}+\frac{d_2+1}{1-\varepsilon}}{\frac{d_1}{\varepsilon}+\frac{d_2}{1-\varepsilon}}$$

It can be shown that whenever $d_1 > d_2$ the elasticity of substitution is decreasing over the support $[0, 1]$ and vice versa. This means that the larger d_1 compared to d_2, that is the more skewed to the left tasks are, the higher the elasticity of substitution in tasks closed to 0 compared to the elasticity of substitution in tasks closed to 1.

4 Implications

The assignment model proposed in this paper provides microfoundations to production functions. By deriving the shape of the aggregate production function from the assignment of heterogenous workers to heterogenous tasks, the model provides a way to evaluate how stringent assumptions about the type of production functions or technological change are by comparing the implied distribution of jobs and its evolution over time to observations of the distribution of jobs and its evolution over time.

For instance, the workhorse model in the skill-biased technical change literature (see Katz and Murphy (1992) or Acemoglu (2002)) assumes that i) aggregate output is produced with a CES technology and ii) technical change can only alter the efficiencies of skilled and unskilled labor over time, i.e. σ, the elasticity of substitution parameter, is constant over time. The assignment model developed in this paper can be used to evaluate how realistic these assumptions are. Assumption i) implies that $\theta_1 = \theta_2 = \theta$ and therefore that the distribution of jobs is symmetric around task 0.5. Assumption ii) implies that only the efficiency parameters r_k may change over time and hence that θ_1, θ_2 and θ are constant over time. Assumption ii) has two implications. Implication 1 is that the distribution of tasks does not change over time. Implication 2 is that technological changes affect the production parameters m_j and n_j and assignment a_j so that $\frac{dm_j}{dt} + a_j \frac{dn_j}{dt} + n_j \frac{da_j}{dt} = 0$. These very specific changes are very unlikely to have occurred in real data and recent evidence provided by Autor et al. (2006) about the job polarization in the US economy is in sharp contrast with implication 1. Moreover, as pointed out early in the literature (see Brynjolfsson (1995), Bresnahan (1999) and Bresnahan et al. (2002) and van Reenen and Caroli (2001) for instance), technical changes are usually complemented with organizational changes so that some tasks may disappear, for instance "Tellers" in the Banking sector after the introduction of ATM's (see Hunter et al. (2001)), and new tasks are created, "Customer Services Representative" in the banking sector, which also challenges implication 1. A more likely type of technical changes would not restrict changes in m_j and n_j and the resulting assignment a_j and allow for changes in the distribution of tasks altogether, i.e. $\frac{dm_j}{dt} + a_j \frac{dn_j}{dt} + n_j \frac{da_j}{dt} = \frac{dd_j}{dt}$.

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